→ read handout, page 2

Taylor / Maclaurin Series / Polynomials

For the next to Examples,

for the given function $y = f(x)$ and center x_0, find:

① the N^{th} order Taylor polynomial of $y = f(x)$ at x_0 for $N = 0, 1, 2, 3, 4$. So find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$.

② Find a general formula for the n^{th} Taylor coefficient of $y = f(x)$ at x_0. So find C_n.

③ the Taylor series of $y = f(x)$ at x_0, in closed form using the sigma notation. So find $P_\infty(x)$.

Taylor Example 1 $\quad f(x) = \frac{1}{1-x}$ and $x_0 = 0$.

① $$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f^{(1)}(0)\,x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(N)}(0)}{N!}x^N$$

=

$$\left.\begin{aligned}
f^{(0)}(x) \overset{ie}{=} f(x) &= (1-x)^{-1} \\
f'(x) &= +(1-x)^{-2} \\
f^{(2)}(x) &= 2(1-x)^{-3} \\
f^{(3)}(x) &= 2\cdot 3(1-x)^{-4} \\
f^{(4)}(x) &= 2\cdot 3\cdot 4(1-x)^{-5}
\end{aligned}\right] \Rightarrow
\begin{aligned}
f(0) &= 1 \\
f'(0) &= 1 \\
f^{(2)}(0) &= 2 \\
f^{(3)}(0) &= 3! \\
f^{(4)}(0) &= 4!
\end{aligned}$$

=

$P_0(x) = 1$

$P_1(x) = 1 + 1\cdot x$

$P_2(x) = 1 + 1x + \frac{2}{2!}x^2$

$P_3(x) = 1 + 1\cdot x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3$

$P_4(x) = 1 + 1\cdot x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4$

So

$P_0(x) = 1$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + x^2$

$P_3(x) = 1 + x + x^2 + x^3$

$P_4(x) = 1 + x + x^2 + x^3 + x^4$.

(2) $c_n = \dfrac{f^{(n)}(x_0)}{n!} = \dfrac{f^{(n)}(0)}{n!}$

$f(x) = (1-x)^{-1} = 0!\,(1-x)^{-1}$

$f'(x) = (1-x)^{-2} = 1!\,(1-x)^{-2}$

$f^{(2)}(x) = 2(1-x)^{-3} = 2!\,(1-x)^{-3}$

$f^{(3)}(x) = 2\cdot 3\,(1-x)^{-4} = 3!\,(1-x)^{-4}$

$f^4(x) = \underbrace{2\cdot 3\cdot 4\,(1-x)^{-5}} = 4!\,(1-x)^{-5}$

already did
do you see the pattern?

$\Rightarrow$

$f^{(0)}(0) = 1 = 0!$

$f^{(1)}(0) = 1!$

$f^{(2)}(0) = 2!$

$f^{(3)}(0) = 3!$

$f^{(4)}(0) = 4!$

do you see the pattern?

So $f^{(n)}(0) = n!$ for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots$

$c_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{n!}{n!} = 1$ for $n = 0, 1, 2, \ldots$

$\boxed{c_n = 1 \quad \text{for } n = 0, 1, 2, \ldots}$

(3) $P_\infty(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \underset{\text{here}}{\overset{x_0 = 0}{=}} \sum_{n=0}^{\infty} \boxed{\frac{f^{(n)}(0)}{n!}}\, x^n = \sum_{n=0}^{\infty} 1\cdot x^n$

$\boxed{P_\infty(x) = \displaystyle\sum_{n=0}^{\infty} x^n}$

note $1 = x^0$

compare

In open form, $P_\infty(x) = \underbrace{1}_{\hookrightarrow x^0} + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$

Taylor Example 2 $\quad f(x) = \sin x \quad$ and $\quad x_0 = \pi$

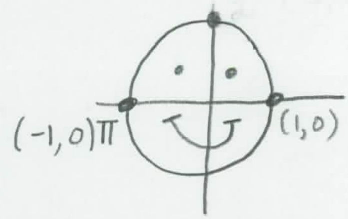
in calculus we always work in radians not degrees ! why ?

① $$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{N} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

starts to repeat

$$\begin{aligned} f^{(0)}(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned} \quad \Rightarrow \quad \begin{aligned} f^{(0)}(\pi) &= 0 \\ f^{(1)}(\pi) &= -1 \\ f^{(2)}(\pi) &= 0 \\ f^{(3)}(\pi) &= +1 \\ f^{(4)}(\pi) &= 0 \end{aligned}$$

starts to repeat



helpful

$$P_4(x) = f^{(0)}(\pi) + f^{(1)}(\pi)(x-\pi) + \frac{f^{(2)}(\pi)}{2!}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4$$

$$\begin{aligned} P_0(x) &= 0 && = 0 \\ P_1(x) &= 0 - (x-\pi) && = -(x-\pi) \\ P_2(x) &= 0 - (x-\pi) + 0 && = -(x-\pi) \\ P_3(x) &= 0 - (x-\pi) + 0 + \tfrac{1}{3!}(x-\pi)^3 && = -(x-\pi) + \tfrac{1}{3!}(x-\pi)^3 \\ P_4(x) &= 0 - (x-\pi) + 0 + \tfrac{1}{3!}(x-\pi)^3 + 0 && = -(x-\pi) + \tfrac{1}{3!}(x-\pi)^3 \end{aligned}$$

Why they are called N^{th} order Taylor polynomials instead of N^{th} degree Taylor polynomials

[The degree of the N^{th} order Taylor polynomial] $\leq N$.

② $$c_n = \frac{f^{(n)}(x_0)}{n!} = \frac{f^{(n)}(\pi)}{n!}$$

$$f^{(n)}(\pi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pm 1 & \text{if } n \text{ is odd} \end{cases}$$

so

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{\pm 1}{n!} & \text{if } n \text{ is odd} \end{cases}$$

yek... lets do ③ first

11.58

(3) From (1) we see

$$P_\infty(x) = -(x-\pi)^1 + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \frac{1}{9!}(x-\pi)^9 \pm \cdots$$

$\Rightarrow P_\infty(x)$ looks like $\underset{\text{over odd } n}{\sum} (\pm 1) \frac{1}{n!}(x-\pi)^n$ ← clean up!

Note

$$\{2n\}_{n=0}^{\infty} = \{ \underset{n=0}{0}, \underset{n=1}{2}, \underset{n=2}{4}, \underset{n=3}{6}, \ldots \}$$ ← even #'s

$$\{2n+1\}_{n=0}^{\infty} = \{1, 3, 5, 7, \ldots\}$$ ← odd #'s

$$\Rightarrow P_\infty(x) = \sum_{n=\boxed{?}}^{\infty} (\pm 1) \frac{1}{(2n+1)!}(x-\pi)^{2n+1}$$

want to "start" with $(2n+1) = 1$ so want to start with $n = 0$

$$\Rightarrow P_\infty(x) = \sum_{n=0}^{\infty} (\pm 1) \frac{1}{(2n+1)!}(x-\pi)^{2n+1}$$

how should this look? +1 or −1? Let's make a chart to see

n	$2n+1$	+1 or −1		
0	1	−1	$= (-1)^{\text{odd \#}}$	$= (-1)^{n+1}$
1	3	1	$= (-1)^{\text{even \#}}$	$= (-1)^{n+1}$
2	5	−1	$= (-1)^{\text{odd \#}}$	$= (-1)^{n+1}$
3	7	1	$= (-1)^{\text{even \#}}$	$= (-1)^{n+1}$
⋮	⋮			⋮

$$\Rightarrow \boxed{P_\infty(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!}(x-\pi)^{2n+1}} \text{ or } \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

(2) Find a general formula for $c_n \equiv \dfrac{f^{(n)}(x_0)}{n!}$.

We have already noted:

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{\pm 1}{n!} & \text{if } n \text{ is odd} \end{cases}.$$

In (3) we found

$$P_\infty(x) = \sum_{k=0}^{\infty} c_{k} (x-\pi)^{k}$$

these always match up.

$$P_\infty(x) = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^{n+1}}{(2n+1)!}} (x-\pi)^{2n+1}$$

here must be c_{2n+1}

So:

even $\quad c_{2n} = 0 \qquad n = 0, 1, 2, \ldots \quad (\text{so } 2n = 0, 2, 4, \ldots)$

odd. $\quad c_{2n+1} = \dfrac{(-1)^{n+1}}{(2n+1)!} \qquad n = 0, 1, 2, \ldots \quad (\text{so } 2n+1 = 1, 3, 5, \ldots)$

Theorem If f has a power series representation (expansion) at $x_0 = a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then its coefficients are

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

In short, if a function can be represented by a power series, then that power series is the function's Taylor series!

→ read handout pages 3 and 4.

<u>Revisit</u> Taylor Example 2

The Taylor series of

$$f(x) = \sin x$$

about $x_0 = \pi$ is

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1} .$$

We already did.

Now show that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}$$

for all $x \in (-\infty, \infty)$.

<u>Sol'n</u> Know

$$f(x) = P_N(x) + R_N(x) .$$

We want to show that

$$\lim_{N\to\infty} R_N(x) = 0$$

or equivalently

$$\lim_{N\to\infty} |R_N(x)| = 0$$

for all $-\infty < x < \infty$.

So fix an $x \in (-\infty, \infty)$. The BIG Theorem says that for some c between x and π x_0 :

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x-\pi)^{N+1}$$

<u>Look back at</u> Taylor Example 2

$$f(x) = \sin x$$

$f^{(N+1)}(x)$ is either $\cos x$, $-\cos x$, $\sin x$, or $-\sin x$,

so $|f^{(N+1)}(c)| \le 1$.

$$|R_N(x)| = |f^{(N+1)}(c)| \frac{1}{(N+1)!} |x-\pi|^{N+1}$$

$$\downarrow$$

$$\leq 1 \frac{|x-\pi|^{N+1}}{(N+1)!}$$

Also see homework assignment #10.2 #27.

$\Rightarrow$

$$|R_N(x)| \leq \frac{|x-\pi|^{N+1}}{(N+1)!} \xrightarrow[\text{why?}]{N\to\infty} 0$$

by n^{th} term test.

by ratio test

$$\Downarrow$$

$$\sum \frac{|x-\pi|^{N+1}}{(N+1)!}$$

☒ absolute convergent

☐ ~~conditionally convergent~~ b/c $\frac{|x-\pi|^{N+1}}{(N+1)!} > 0$

☐ divergent

<u>Ratio Test</u> $\rho = \lim_{N\to\infty} \frac{|x-\pi|^{N+2}}{(N+2)!} \frac{(N+1)!}{|x-\pi|^{N+1}} = |x-\pi| \lim_{N\to\infty} \frac{1}{N+2} = |x-\pi| \cdot 0 < 1$

So, since for all $x \in (-\infty, \infty)$

$$\lim_{N\to\infty} R_N(x) = 0,$$

we know that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}$$

for all $x \in (-\infty, \infty)$.

<u>Aside</u> Useful fact. For all $x \in \mathbb{R}$:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 = \lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = \lim_{n\to\infty} \frac{|x|^n}{n!}$$

<u>why?</u> b/c even more is true, namely $\sum_n \frac{x^n}{n!}$ is abs. conv.

by ratio test b/c $\rho = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

11.62

Example 2 Let $x_0 = 0$ and

$$f(x) = \ln(1+x)$$

(2a) Find the Taylor series for $y = f(x)$ about $x_0 = 0$.

$f(x) = \ln(1+x) \longrightarrow f^{0}(0) = f(0) = \ln 1 = 0$

$f^{(1)}(x) = (1+x)^{-1}$

$f^{(2)}(x) = -(1+x)^{-2}$

$f^{(3)}(x) = 2(1+x)^{-3}$

$f^{(4)}(x) = -2\cdot 3(1+x)^{-4}$

$f^{(5)}(x) = 2\cdot 3\cdot 4(1+x)^{-5}$

⋮ we see the pattern

need to compute $f^{(0)}(0)$ separately

$\boxed{n \geq 1}$

$\boxed{\text{for } n \geq 1} \longrightarrow f^{(n)}(x) = (-1)^{(n-1)}(n-1)!\,(1+x)^{-n} \Rightarrow f^{(n)}(0) = (-1)^{(n-1)}(n-1)!$

So $$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

need $n \geq 1$ to use so start Σ at $n=$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n$$

$$\Rightarrow \boxed{P_\infty(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n}$$

(2b) Remark. If we apply methods from power series (§ 10.8)

then we see that the interval of convergence

of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ is $(-1, +1]$. ← (Ratio Test)

This you can do.

(2c) Fact

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad \underbrace{\text{for } x \in (-1,+1]}_{\text{i.e. } -1 < x \leq 1}.$$

But the proof of this Fact is very hard for $-1 < x < -\frac{1}{2}$.

So let's do an easier problem.

(2d) Show that for $-\frac{1}{4} \leq x \leq \frac{5}{8}$.

$$\underbrace{\ln(1+x)}_{f(x)} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n}_{\text{Taylor series for } f(x) \text{ about } x_0 = 0}.$$

Sol'n.

Here the N^{th} order Taylor polynomial is

$$P_N(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} x^n.$$

Know

$$f(x) = P_N(x) + R_N(x).$$

Fix $x \in \left[-\frac{1}{4}, \frac{5}{8}\right]$. We want to show

$$\lim_{N\to\infty} R_N(x) = 0$$

or equivalently

$$\lim_{N\to\infty} |R_N(x)| = 0.$$

$\boxed{\text{BIG Theorem}}$ for some c between x and 0 $\leftarrow x_0$

$$|R_N(x)| = \frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1}$$

$$\Rightarrow \quad |R_N(x)| = |f^{(N+1)}(c)| \frac{|x|^{N+1}}{(N+1)!}$$

ALREADY DID

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Let $n = N+1$

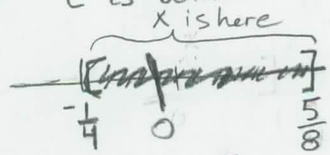
$x = c$

$$= \left| \frac{(-1)^N N!}{(1+c)^{N+1}} \right| \frac{|x|^{N+1}}{(N+1)!}$$

$$= \frac{1}{N+1} \cdot \left[\frac{|x|}{|1+c|} \right]^{N+1}$$

Given $-\frac{1}{4} \le x \le \frac{5}{8}$

Know c is between x & 0



So $-\frac{1}{4} \le c \le \frac{5}{8}$

$\rightarrow$ add 1 across

So $\frac{3}{4} \le 1+c \le \frac{13}{8}$

$$\le \frac{1}{N+1} \left[\frac{5/8}{3/4} \right]^{N+1}$$

$$\boxed{\left(\frac{5}{6}\right)^{N+1} \le 1}$$

$$= \frac{1}{N+1} \left(\frac{5}{6} \right)^{N+1}$$

$$\le \frac{1}{N+1} \xrightarrow{N \to \infty} 0$$

So $\lim\limits_{N \to \infty} R_N(x) = 0$ if $-\frac{1}{4} \le x \le \frac{5}{8}$.

So $\ln(1+x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ if $-\frac{1}{4} \le x \le \frac{5}{8}$