Partial Fractions Handout - read thru page 1

Ex 0 Partial Factions Handout - "A common mistake", bottom page 2.

Ex 1 Write out the form of the Partial Fraction Decomposition (PFD) :

Would make a good fill-in-the-blank problem

$$\frac{x^3-4x^2+2}{(x-1)^4(x^2-16)x^2(x^2-3x+4)^3}$$

bigger bottom? degree num. $= 3 < 4+2+2+2(3) = 14 =$ degree den.

→ yes.

go do ① $(x-1)^4 = (\text{linear term})^4$ contribute 4 factors

② $(x^2-16) = [(x-4)(x+4)] = (\text{linear term})^1(\text{another linear term})^1$ each contribute 1 factor

$b^2-4ac = 0^2-4(1)(-16) \not< 0 \Rightarrow$ reducible quadratic

③ $x^2 = (x-0)^2 = (\text{linear term})^2$ contribute 2 factors

$b^2-4(a)(c) = 0^2-4(1)(0) = 0 \not< 0 \Rightarrow$ reducible quadratic

④ $(x^2-3x+4)^3 = (\text{irreducible quadratic})^3$ contribute 3 factors.

$b^2-4ac = 9-4(1)(4) < 0 \Rightarrow$ irreducible quadratic

so

$$\frac{x^3-4x^2+2}{(x-1)^4(x^2-16)x^2(x^2-3x+4)^3} =$$

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{E}{x-4}+\frac{F}{x+4}+\frac{G}{x}+\frac{H}{x^2}$$

$$+\frac{Ix+J}{x^2-3x+4}+\frac{Kx+L}{(x^2-3x+4)^2}+\frac{Mx+N}{(x^2-3x+4)^3}$$

Ex 2 $\displaystyle\int \frac{x^3-4x-1}{x(x-1)^3}\,dx$

- bigger bottom? Yes/No degree num. $= 3 < 4$ degree dem.
- $x = (x-0)^1 = (\text{linear term})^1$ ← contribute 1 factor
- $(x-1)^3 = (\text{linear term})^3$ ← contribute 3 factors

$$\frac{x^3-4x-1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} = \frac{A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx}{x(x-1)^3}$$

get common den.

plug in the zeros of the linear terms

equate numerators

$$x^3-4x-1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx \quad \begin{cases} x=0 \Rightarrow -1 = -A \Rightarrow \boxed{A=1} \\ x=1 \Rightarrow \boxed{-4 = D} \end{cases}$$

$$x^3-4x-1 = A(x^3-3x^2+3x-1) + B(x^3-2x^2+x) + C(x^2-x) + Dx$$

Equate coefficients - long way (multiply out & collect like terms) or short way.

$$x^3-4x-1 = (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+D)x - A$$

$$\begin{aligned}
\Rightarrow x^3 &: 1 = A+B \xrightarrow{\text{solve for}} 1 = 1+B \Rightarrow \boxed{B=0} \\
x^2 &: 0 = -3A-2B+C \longrightarrow 0 = -3(1)-2(0)+C \Rightarrow \boxed{C=3} \\
x^1 &: -4 = 3A+B-C+D \\
\text{constant} &: -1 = -A
\end{aligned}$$

solve for constants

So

$$\int \frac{x^3-4x-1}{x(x-1)^3}\,dx = \int\left[\frac{1}{x} + \frac{0}{x-1} + \frac{3}{(x-1)^2} + \frac{-4}{(x-1)^3}\right]dx$$

$$= \int \frac{dx}{x} + 3\int (x-1)^{-2}\,dx - 4\int (x-1)^{-3}\,dx$$

$$= \ln|x| + \frac{3(x-1)^{-1}}{-1} - 4\,\frac{(x-1)^{-2}}{-2} + K$$

we already used "C"

$$= \boxed{\ln|x| - \frac{3}{x-1} + \frac{2}{(x-1)^2} + K}$$

Ex 3 $\displaystyle\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$

- bigger bottoms? Yes / No [degree num] = 3 < 4 = [degree den].
- $x^4+x^2 = x^2(x^2+1)$

$x^2=(x-0)^2 = (\text{linear term})^2$ contribute 2 factors

$b^2-4ac = 0^2-4(1)(1)<0 \Rightarrow (x^2+1)^1 = (\text{irred. quad.})^1$ 1 factor

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2}{x^2(x^2+1)}$$

should always be the same

$$\Rightarrow 5x^3-3x^2+2x-1 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$$

$\boxed{x=0} \Rightarrow \boxed{-1=B}$

Equate coefficients (short way) ← do on board for them

x^3	:	$5 = A + C$
x^2	:	$-3 = B + D$
x	:	$\boxed{2 = A}$
constant	:	$-1 = B$

$\Rightarrow \boxed{C=3}$

$\Rightarrow \boxed{D=-2}$

So

$$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2} = \int \left[\frac{2}{x}+\frac{-1}{x^2}+\frac{3x-2}{x^2+1}\right]dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= 2\int\frac{dx}{x} - 1\int x^{-2}\,dx + 3\,\frac{1}{2}\int \underset{u^{-1}}{(x^2+1)^{-1}}\,\underbrace{(2x\,dx)}_{du} - 2\int\frac{dx}{x^2+1}$$

$$= \boxed{2\ln|x| - \frac{x^{-1}}{-1} + \frac{3}{2}\ln|x^2+1| - 2\tan^{-1}x + K.}$$

Ex 4

$$\int \frac{5x^3-3x^2+7x-3}{\underbrace{(x^2+1)^2}}\,dx$$

• bigger bottoms? Yes / No

(degree num) = 3 < 4 = (degree den.)

$\hookrightarrow b^2-4ac = 0^2-4(1)(1) < 0 \Rightarrow$ (irred. quad.)^2 ← contribute 2 factors.

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)^2}$$

$$\Rightarrow 5x^3-3x^2+7x-3 = (Ax+B)(x^2+1)+(Cx+D) \quad \boxed{x=?}$$

no go bcs no linear terms

x^3 : $\boxed{5 = A}$

x^2 : $\boxed{-3 = B}$ $\Rightarrow$ $\boxed{C=2}$

x : $7 = A + C$

constant : $-3 = B + D$ $\Longrightarrow$ $\boxed{D=0}$

$$\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \int\left[\frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}\right]dx$$

$$= 5\,\frac{1}{2}\int \underset{u=x^2+1}{(x^2+1)^{-1}}\ \underset{du=2x\,dx}{(2x\,dx)} - 3\int\frac{dx}{x^2+1} + \int \underset{u=x^2+1}{(x^2+1)^{-2}}\ \underset{du=2x\,dx}{(2x\,dx)}$$

$$= \frac{5}{2}\ln|x^2+1| - 3\tan^{-1}x + \frac{(x^2+1)^{-1}}{-1} + K$$

why?

$$= \boxed{\frac{5}{2}\ln(x^2+1) - 3\tan^{-1}x - \frac{1}{x^2+1} + K.}$$

Read. Partial Fractions Handout - page 2

Ex 5 $\displaystyle\int \frac{3x+8}{x+1}\,dx$

• bigger bottoms? Yes / No [degree num] = 1 = 1 = [degree den.]

need strictly bigger bottoms.

→ So need to do long division - but this one is easy enough that we can fake long division

$$\frac{3x+8}{x+1} \underset{\text{fake L.D.}}{\overset{\text{trick}}{=}} \frac{\boxed{?}\,(x+1)+\boxed{?}}{(x+1)} = \frac{3(x+1)+\boxed{?}}{x+1} = \frac{3(x+1)+5}{(x+1)} = 3+\frac{5}{x+1}$$

$$\int \frac{3x+8}{x+1}\,dx = \int\left[3+\frac{5}{x+1}\right]dx = 3\int dx + 5\int\frac{dx}{x+1} = \boxed{3x+5\ln|x+1|+C}$$

Ex 6 $\displaystyle\int \frac{3x^2+8}{x+1}\,dx$ • bigger bottoms? Yes / No ⟹ do long division

[degree num] = 2 > 1 = [degree den]

$$\begin{array}{r} 3x \quad + -3 \\ x+1 \,\overline{)\, 3x^2 + 0x + 8} \\ -(3x^2 + 3x) \\ \hline -3x + 8 \\ -(-3x-3) \\ \hline 11 \end{array}$$

$$\Longrightarrow \frac{3x^2+8}{x+1} = (3x-3) + \frac{11}{x+1}$$

$$\int \frac{3x^2+8}{x+1}\,dx = \int\left(3x-3+\frac{11}{x+1}\right)dx = \boxed{\frac{3x^2}{2} - 3x + 11\ln|x+1| + C}$$

Ex 7 $\displaystyle\int \frac{dx}{4x^2-12x+34}$

• bigger bottoms? Yes

• $b^2-4ac = 12^2-4(4)(34) = -400 < 0 \Rightarrow$

$4x^2-12x+34 = (\text{irred quad})^1$

• $\dfrac{1}{4x^2-12x+34} = \dfrac{Ax+B}{4x^2-12x+34}$ ⟹ $A=0$ & $B=1$

⟹ already in its PFD.

$$\int\frac{dx}{4x^2-12x+34} \underset{\text{square}}{\overset{\text{Complete}}{=}} \int\frac{dx}{(2x-3)^2+25} \longleftarrow \boxed{\begin{array}{l} u=2x-3 \\ du = 2dx \end{array}}$$

$$= \frac{1}{2}\int\frac{(2\,dx)}{(2x-3)^2+5^2} = \frac{1}{2}\int\frac{du}{u^2+a^2} = \frac{1}{2}\frac{1}{a}\tan^{-1}\frac{u}{a}+C = \boxed{\frac{1}{10}\tan^{-1}\frac{2x-3}{5}+C}$$

$\boxed{a=5}$

8.21