Ex 0 $\displaystyle\int \frac{dx}{x^2+25}$

Oh no ... I know it's a $\tan^{-1}$ but I forget the formula

that $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$.

Atlas ... I can use trig substitution to save me!

It's a $\boxed{u^2 + a^2}$ where $u = x$ and $a = 5$ so use $u = a\tan\theta$.

$$\boxed{\begin{aligned} &x = 5\tan\theta \Rightarrow \tan\theta = \frac{x}{5} \\ &\Downarrow \\ &dx = 5\sec^2\theta\, d\theta \\ &x^2 + 25 = 25\tan^2\theta + 25 = 25(\tan^2\theta + 1) = 25\sec^2\theta \end{aligned}}$$

$$\int \frac{dx}{x^2+25} = \int \frac{5\sec^2\theta\, d\theta}{25\sec^2\theta} = \frac{1}{5}\int d\theta = \frac{1}{5}\theta + C = \boxed{\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C}$$

Remark § 8.1, page 512, formula 24-28 (which you do not need to memorize) were all obtained by trig substitution.

Ex 0 $\displaystyle\int \frac{dx}{x^2+25}$ revisited

Correct trig substitution is $x = 5\tan\theta$ .... all worked

A Wrong trig substitution is $x = 5\sin\theta$ ... let's see what happens.

$x = 5\sin\theta$

$\underbrace{x^2+25}_{\text{in denominator}} = (5\sin\theta)^2 + 25 = 25\sin^2\theta + 25 = 25(\underline{\sin^2\theta + 1})$

here is the trouble .... we know:

- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$

but these 3 do not help us with $\underline{\sin^2\theta + 1}$

he helped when we used the correct trig substitution! (bs)

Ex 1 $\displaystyle\int \frac{x^3\,dx}{\sqrt{4-x^2}}$

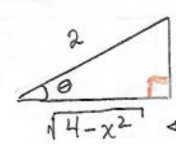
$4-x^2 = (2)^2-(x)^2 \Rightarrow a=2$ and $u=x$ and use $u=a\sin\theta$

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \\ \sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = 2\sqrt{1-\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2|\cos\theta| \overset{\text{why?}}{=} 2\cos\theta \end{aligned}}$$

$$\int \frac{x^3\,dx}{\sqrt{4-x^2}} = \int \frac{(2\sin\theta)^3\,[2\cos\theta\,d\theta]}{2\cos\theta} = 8\int \sin^3\theta\,d\theta$$

$$\boxed{\begin{aligned} s &= \cos\theta \\ ds &= -\sin\theta\,d\theta \end{aligned}} \quad \boxed{\cancel{\begin{aligned} t &= \sin\theta \\ dt &= \cos\theta\,d\theta \end{aligned}}}$$

$x = 2\sin\theta \Downarrow \sin\theta = \frac{x}{2} \Rightarrow$



reference triangle

bs

$\Rightarrow \cos\theta = \frac{\sqrt{4-x^2}}{2}$

$\cos\theta = \frac{(4-x^2)^{1/2}}{2}$

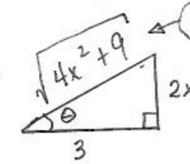
$$= -8\int \sin^2\theta\,[-\sin\theta\,d\theta] = -8\int (1-\cos^2\theta)\,[-\sin\theta\,d\theta]$$

$$= -8\int (1-s^2)\,ds = 8\int (s^2-1)\,ds = \frac{8s^3}{3} - 8s + C = \frac{8}{3}\cos^3\theta - 8\cos\theta + C$$

$$= \frac{8}{3}\left[\frac{(4-x^2)^{1/2}}{2}\right]^3 - 8\left[\frac{(4-x^2)^{1/2}}{2}\right] + C = \boxed{\frac{(4-x^2)^{3/2}}{3} - 4(4-x^2)^{1/2} + C}$$

Ex 2 $\displaystyle\int \frac{dx}{(4x^2+9)^2} = \int \frac{dx}{[(2x)^2+(3)^2]^2} = \int \frac{\frac{3}{2}\sec^2\theta\,d\theta}{(9\sec^2\theta)^2} =$

$u = 2x$, $a=3$, $u^2+a^2 \Rightarrow u = a\tan\theta$ $\quad \tan\theta = \frac{2x}{3} \Rightarrow$



bs

$2x = 3\tan\theta$

$2\,dx = 3\sec^2\theta\,d\theta$

$(2x)^2+(3)^2 = (3\tan\theta)^2+3^2 = 9\tan^2\theta+9 = 9(\tan^2\theta+1) = 9\sec^2\theta$

$$= \frac{3}{2}\cdot\frac{1}{9}\cdot\frac{1}{9}\int\frac{d\theta}{\sec^2\theta} = \frac{1}{54}\int\cos^2\theta\,d\theta = \frac{1}{54}\cdot\frac{1}{2}\int(1+\cos 2\theta)\,d\theta = \frac{1}{108}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

Double Angle Formula

$$= \frac{\theta}{108} + \frac{1}{108}\cdot\frac{1}{2}\,2\sin\theta\cos\theta + C = \frac{1}{108}\tan^{-1}\left(\frac{2x}{3}\right) + \frac{1}{108}\left(\frac{2x}{\sqrt{4x^2+9}}\right)\left(\frac{3}{\sqrt{4x^2+9}}\right) + C$$

$$= \boxed{\frac{1}{108}\tan^{-1}\left(\frac{2x}{3}\right) + \frac{x}{18(4x^2+9)} + C}$$

Ex 3 $\displaystyle\int_{x=-2\sqrt{2}-1}^{x=-3} \frac{x\,dx}{\sqrt{x^2+2x-3}} \underset{\text{square}}{\overset{\text{complete}}{=}} \int_{x=-2\sqrt{2}-1}^{x=-3} \frac{x\,dx}{\sqrt{(x+1)^2-4}}$

$u = x+1, \quad a = 2, \quad u^2 - a^2 \Rightarrow u = a\sec\theta$

$\boxed{x+1 = 2\sec\theta}$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2+2x-3} = \sqrt{(x+1)^2-4} = \sqrt{4\sec^2\theta-4} = \sqrt{4(\sec^2\theta-1)} = 2\sqrt{\sec^2\theta-1}$

$= 2\sqrt{\tan^2\theta} = 2|\tan\theta| \overset{?}{=} \pm 2\tan\theta = -2\tan\theta$

why do we have to worry about $\pm$ here?

Know: $-2\sqrt{2}-1 \le x \le -3 \overset{??}{\Longrightarrow}$ what about θ ... well, we know $\sec\theta = \frac{x+1}{2}$ so $\theta = \sec^{-1}\left(\frac{x+1}{2}\right)$

$\Downarrow$ so

$-2\sqrt{2} \le x+1 \le -2 \Rightarrow -\sqrt{2} \le \frac{x+1}{2} \le -1 \Rightarrow -\sqrt{2} \le \sec\theta \le -1$

So: [$\sec\theta$ is negative] and [$\theta = \sec^{-1}\frac{x+1}{2}$ is in $\underbrace{1^{st}/2^{nd}}_{\text{why?}}$ Quadrant]

So: θ is in 2^{nd} Quad so $\tan\theta$ is negative

limits of integration

$x = -2\sqrt{2}-1 \Rightarrow \sec\theta = \frac{x+1}{2} = -\sqrt{2} = \frac{1}{\cos\theta} \Rightarrow \theta = \frac{3\pi}{4}$

$x = -3 \Rightarrow \sec\theta = \frac{x+1}{2} = -1 = \frac{1}{\cos\theta} \Rightarrow \theta = \pi$

recall, θ is in 2^{nd} Quad.

$$\int_{x=-2\sqrt{2}-1}^{x=-3} \frac{x\,dx}{\sqrt{x^2+2x-3}} = \int_{\theta=\frac{3\pi}{4}}^{\theta=\pi} \frac{(2\sec\theta-1)\,[\cancel{2}\sec\theta\tan\theta\,d\theta]}{-\cancel{2}\tan\theta}$$

$$= \int_{\theta=3\pi/4}^{\theta=\pi} (-2\sec^2\theta + \sec\theta)\,d\theta = \left[-2\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_{\theta=\frac{3\pi}{4}}^{\theta=\pi}$$

bs

$$= \left[-2\tan\pi + \ln|\sec\pi + \tan\pi|\right] - \left[-2\tan\tfrac{3\pi}{4} + \ln\left|\sec\tfrac{3\pi}{4} + \tan\tfrac{3\pi}{4}\right|\right]$$

$$= \left[-2(0) + \ln|-1+0|\right] - \left[-2(-1) + \ln|-\sqrt{2}-1|\right]$$

$$= 0 + \ln 1 - 2 - \ln|(-1)(1+\sqrt{2})| = \boxed{-2 - \ln(1+\sqrt{2})}$$