

Chapter 9 Miscellaneous Problems

1. Let  $x = u^2$ .

2.  $\ln|1 + \tan t| + C$

4.  $-\tan^{-1}(\csc x) + C$ , using  $u = \csc x$  if you wish.

6.  $\csc^4 x = (1 + \cot^2 x)(\csc^2 x) = \csc^2 x + \cot^2 x \csc^2 x$ .

7. Let  $u = x$ ,  $dv = \tan^2 x dx = (\sec^2 x - 1) dx$ ; then  
 $du = dx$  and  $v = -x + \tan x$ .

8. First write

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Integrate what you can, and then integrate  $x^2 \cos 2x$  by parts:  
 Let  $u = x^2$  and  $dv = \cos x dx$ . This leads to the  
 integration of  $x \sin 2x$ , which also works well by parts. You  
 should get

$$\frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x \cos 2x - \frac{1}{8} \sin 2x + C.$$

9. Let  $u = 2 - x^3$ .

10. Let  $x = 2 \tan u$ . The integrand becomes  $\sec u$ , and the  
 antiderivative is

$$\begin{aligned} \ln|\sec u + \tan u| + C &= \ln|\frac{1}{2}(x^2 + 4)^{1/2} + \frac{1}{2}x| + C_1 \\ &= \ln|x + (x^2 + 4)^{1/2}| + C. \end{aligned}$$

11. Let  $x = 5 \tan \xi$ . The integrand becomes  $25(\sec^3 \xi - \sec \xi)$ .

12. Let  $u = \sin x$ . The integrand becomes  $(4 - u^2)^{1/2}$ . Then let  
 $u = 2 \sin \xi$ . The integrand then becomes  $4 \cos^2 \xi$ , and the  
 rest is easy. The answer may be written in the form

$$2 \sin^{-1}\left(\frac{1}{2} \sin x\right) + \frac{1}{2}(\sin x)(4 - \sin^2 x)^{1/2} + C.$$

13. Write  $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$ , then apply  
Formula 17 from the endpapers.

14. First write  $x^2 + x + 1$  as  $\frac{3}{4}(u^2 + 1)$  where

$$u = \frac{2x + 1}{\sqrt{3}}.$$

Then let  $u = \tan \xi$ , and the integrand becomes  $\frac{3}{4} \sec^3 \xi$ .

Apply Formula 28 from the endpapers; the final answer is

$$\begin{aligned} & \frac{1}{4} (2x + 1)(x^2 + x + 1)^{1/2} \\ & + \frac{3}{8} \ln |2x + 1 + 2(x^2 + x + 1)^{1/2}| + C. \end{aligned}$$

15.  $3x^2 - 4x + 11 = \frac{1}{3} \{(3x - 2)^2 + 29\}$ .

16. The integrand is equal to

$$x^2 - 2 + \frac{5}{x^2 + 2}.$$

The antiderivative is

$$\frac{1}{3}x^3 - 2x + \frac{5}{2}\sqrt{2} \tan^{-1}(\frac{1}{2}x\sqrt{2}) + C.$$

17. Use the substitution  $u = \tan \frac{\xi}{2}$ . The integral becomes

$$\begin{aligned} \int \frac{2}{u^2 + 9} du &= \frac{2}{3} \arctan(\frac{u}{3}) + C \\ &= \frac{2}{3} \arctan(\frac{1}{3} \tan \frac{\xi}{2}) + C. \end{aligned}$$

18. The substitution  $x = u^2$  transforms the integrand into

$$2 - \frac{2}{1 + u^2}.$$

The answer is  $2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$ .

19. Use the substitution  $u = \sin x$ .

20. Write the numerator as  $2\cos^2 x - 1$ . The answer is

$$2\sin x - \ln|\sec x + \tan x| + C.$$

21. Let  $u = \ln(\cos x)$ , then compute  $du$ .

22. Let  $x^2 = \sin u$ . The integrand becomes  $\frac{1}{2}\sin^3 u$ , and the antiderivative is

$$\begin{aligned} & \frac{1}{6}(\cos u)(\cos^2 u - 3) + C \\ &= -\frac{1}{6}(x^4 + 2)(1 - x^4)^{1/2} + C. \end{aligned}$$

23. By parts: Let  $u = \ln(1 + x)$ ,  $dv = dx$ , and (!)  $v = x + 1$ .

24. By parts: Let  $u = \sec^{-1} x$ ,  $dv = x dx$ . This leads to the problem of evaluating

$$\int \frac{|x|}{(x^2 - 1)^{1/2}} dx.$$

For  $x > 1$ , we obtain  $\frac{1}{2}\{x^2 \sec^{-1} x - (x^2 - 1)^{1/2}\} + C$ .

For  $x < -1$ , we obtain  $\frac{1}{2}\{x^2 \sec^{-1} x + (x^2 - 1)^{1/2}\} + C$ .

So for  $|x| > 1$ , the antiderivative is

$$\frac{1}{2}\{x^2 \sec^{-1} x - \frac{|x|}{x}(x^2 - 1)^{1/2}\} + C.$$

25. Let  $x = 3\tan u$ , then parts with  $dv = \sec^2 \theta du$

26. The substitution  $x = 2\sin u$  yields

$$\int 4\sin^2 u du = 2 \int (1 - \cos 2u) du$$

$$= 2u - 2 \sin u \cos u + C$$

$$= 2 \arcsin \frac{x}{2} - \frac{x}{2} (4 - x^2)^{1/2} + C.$$

27.  $2x - x^2 = 1 - (1 - x)^2$ . Now let  $x = 1 - \cos u$ .

28.  $\frac{4x - 2}{x^3 - x} = \frac{2}{x} - \frac{3}{x+1} + \frac{1}{x-1}$ ,

so the antiderivative is

$$2 \ln|x| - 3 \ln|x+1| + \ln|x-1| + C$$

$$= \ln \left| \frac{x^2(x-1)}{(x+1)^3} \right| + C.$$

29. Write the integrand in the form

$$\frac{x^2 + 2 - \frac{4}{2-x^2}}{2-x^2}.$$

Then use Formula 18 from the endpapers, the method of partial fractions, or a trigonometric or hyperbolic substitution.

30.  $\int \frac{\sec x \tan x}{\sec x + \sec^2 x} dx = \int \frac{\tan x}{1 + \sec x} dx$

$$= \int \frac{\sin x}{1 + \cos x} dx = -\ln(1 + \cos x) + C.$$

31. Let  $x = -1 + \tan u$ . The integral becomes

$$\int \frac{-1 + \tan u}{\sec^2 u} du = \int (-\cos^2 u + \sin u \cos u) du,$$

and the rest is routine.

32. The least common multiple of 2, 3, and 4 is 12, so let  
 $u = x^{1/12}$ . This transforms the integral into

$$\int \frac{12u^{12}}{u^3 + 1} du.$$

The result of division of the denominator of the integrand into the numerator yields

$$12(u^9 - u^6 + u^3 - 1 + \frac{1}{u^3 + 1}),$$

and the method of partial fractions yields the decomposition

$$\frac{12}{u^3 + 1} = \frac{4}{u + 1} - \frac{4u - 8}{u^2 - u + 1}.$$

Now proceed much as in the last part of the solution to Problem 8, Section 9.8, on pages 501 - 502. You'll find the antiderivative indicated at the top of this page to be

$$\frac{6}{5}u^{10} - \frac{12}{7}u^7 + 3u^4 - 12u + 4\ln|u + 1|$$

$$- 2\ln|u^2 - u + 1| + 4\sqrt{3}\arctan(\frac{1}{3}\sqrt{3}\{2u - 1\}) + C$$

$$\begin{aligned} &= \frac{6}{5}x^{5/6} - \frac{12}{7}x^{7/12} + 3x^{1/3} - 12x^{1/12} + 4\ln(1 + x^{1/12}) \\ &- 2\ln(1 - x^{1/12} + x^{1/6}) + 4\sqrt{3}\arctan(\frac{1}{3}\sqrt{3}\{2x^{1/12} - 1\}) + C. \end{aligned}$$

$$33. \frac{1}{1 + \cos 2\xi} = \frac{1}{2} \sec^2 \xi.$$

$$34. \int \frac{\sec x}{\tan x} dx = \int \csc x dx = \ln|\csc x - \cot x| + C.$$

$$35. \sec^3 x \tan^3 x = \sec^5 x \tan x - \sec^3 x \tan x.$$

36. By parts: Let  $u = \tan^{-1}x$ ,  $dv = x^2 dx$ . You'll get

$$\begin{aligned}& \frac{1}{3}x^3 \tan^{-1}x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2}\right) dx \\&= \frac{1}{3}x^3 \tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6} \ln(1+x^2) + C.\end{aligned}$$

37. Suggestion: Develop a reduction formula for

$$\int x(\ln x)^n dx$$

by parts; take  $u = (\ln x)^n$  and  $dv = x dx$ . Then apply your formula iteratively to evaluate the given antiderivative. You should find that

$$\int x(\ln x)^n dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2} \int x(\ln x)^{n-1} dx.$$

38. Let  $x = \tan \xi$  to transform the integral into

$$\begin{aligned}\int \csc \xi d\xi &= \ln|\csc \xi - \cot \xi| + C \\&= \ln \frac{(x^2 + 1)^{1/2} - 1}{x} + C.\end{aligned}$$

39. Let  $u = e^x$ , then  $u = \tan z$  to obtain  $\int \sec^3 z dz$ .

40. Note that  $4x - x^2 = 4 - (x - 2)^2$ ; let  $x = 2 + 2 \sin u$ . This substitution transforms the integral into

$$\int 2(1 + \sin u) du = 2(u - \cos u) + C$$

$$= 2 \arcsin \frac{x-2}{2} - (4x - x^2)^{1/2} + C.$$

41. Let  $x = 3 \sec \xi$ .

42. Let  $u = 7x + 1$ ; the answer is  $- \frac{112x + 1}{11760(7x + 1)^{16}} + C$ .

43. Use the method of partial fractions.

44. Divide denominator into numerator, then use the method of partial fractions as in the solution to Problem 32 on page 596. The antiderivative is

$$4x - \frac{2}{3} \ln|x+1| + \frac{1}{3} \ln|x^2 - x + 1|$$

$$- \frac{4}{3} \sqrt{3} \tan^{-1}\left(\frac{1}{3} \sqrt{3} \{2x - 1\}\right) + C.$$

45. Write the integrand as  $\sec^3 x - \sec x$ , then apply the formulas of the endpapers.

46. Here is a quick way to obtain the partial fractions decomposition that you need:

$$\frac{x^2 + 2x + 2}{(x + 1)^3} = \frac{(x + 1)^2 + 1}{(x + 1)^3}.$$

47. The partial fractions decomposition of the integrand is

$$\frac{1}{x+1} + \frac{2}{x^4}.$$

48. The partial fractions decomposition of the integrand is

$$-\frac{1}{x^2 + 1} + \frac{8}{4x + 1},$$

and the antiderivative is therefore

$$2 \ln|4x + 1| - \tan^{-1} x + C.$$

49. The partial fractions decomposition of the integrand has the form

$$\frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} + \frac{D}{(x - 1)^2} + \frac{Ex + F}{(x^2 + x + 1)^2}.$$

The simultaneous equations are

$$\begin{aligned} A + B &= 3, \\ A - B + C + D &= -1, \\ A - C + 2D + E &= 2, \\ -A - B + 3D - 2E + F &= -12, \\ -A + B - C + 2D + E - 2F &= -2, \\ -A + C + D + F &= 1. \end{aligned}$$

Their solution:  $A = 1$ ,  $B = 2$ ,  $C = 1$ ,  $D = -1$ ,  $E = 4$ ,  $F = 2$ . None of the antiderivatives is difficult, and the final answer may be written in the form

$$\ln|x - 1| + \ln(x^2 + x + 1) + \frac{1}{x - 1} - \frac{2}{x^2 + x + 1} + C.$$

50. First,  $x^4 + 4x^2 + 8 = (x^2 + 2)^2 + 4$ . So

$$\int \frac{x}{x^4 + 4x^2 + 8} dx = \int \frac{x}{(x^2 + 2)^2 + 4} dx$$

$$= \frac{1}{4} \int \frac{x}{\{(x^2 + 2)/2\}^2 + 1} dx$$

$$= \frac{1}{4} \arctan\left(\frac{x^2 + 2}{2}\right) + C.$$

51. Use the substitution  $u = \tan \frac{\xi}{2}$ .

52. Let  $x = u^3$  to transform the given integral into

$$\begin{aligned} 3 \int u(1 + u^2)^{3/2} du &= \frac{3}{5} (1 + u^2)^{5/2} + C \\ &= \frac{3}{5} (1 + x^{2/3})^{5/2} + C. \end{aligned}$$

54. Let  $x = u^6$ . The integral becomes

$$6 \int \frac{1}{u^4(1 + u^2)} du.$$

Next let  $u = \tan z$ . The integral above is transformed into

$$\begin{aligned} 6 \int \frac{\sec^2 z}{\tan^4 z \sec^2 z} dz &= 6 \int \cot^4 z dz \\ &= 6 \int (\csc^2 z - 1) \cot^2 z dz \\ &= 6 \int \{\cot^2 z \csc^2 z - (\csc^2 z - 1)\} dz \\ &= 6 \left( -\frac{1}{3} \cot^3 z + \cot z + z \right) + C \\ &= 6 \left( \frac{1}{u} + \arctan u - \frac{1}{3u^3} \right) + C \\ &= 6x^{-1/6} + 6 \tan^{-1}(x^{1/6}) - 2x^{-1/2} + C. \end{aligned}$$

55.  $(\tan z)(\sec^2 z - 1) = (\sec z)(\sec z \tan z) - \tan z$ .

56. Use Formula 39 of the endpapers; alternatively, repeated use of the half-angle formulas and other trigonometric identities may be used to transform the integral into

$$\begin{aligned} & \frac{1}{8} \int \left\{ 1 + \cos 2w - \frac{1}{2}(1 + \cos 4w) - (\cos 2w)(1 - \sin^2 2w) \right\} dw \\ &= \frac{1}{8} \left( \frac{1}{2}w - \frac{1}{4}\sin 2w \cos 2w + \frac{1}{6}\sin^3 2w \right) + C. \end{aligned}$$

57. Because  $e^{2x^2} = e^{x^2}^2$ , let  $u = e^{x^2}$ . Then the pattern will be come clear.

58. Note that

$$\begin{aligned} \frac{\cos^3 x}{(\sin x)^{1/2}} &= \frac{(1 - \sin^2 x)(\cos x)}{(\sin x)^{1/2}} \\ &= (\sin x)^{-1/2} \cos x - (\sin x)^{3/2} \cos x. \end{aligned}$$

So the antiderivative is

$$\frac{2}{5} (\sin x)^{1/2} (5 - \sin^2 x) + C.$$

59. Use integration by parts; choose as  $dv$  the "most difficult part of the integrand that one can actually integrate:"

$$u = x^2, \quad dv = xe^{-x^2} dx.$$

60. Let  $u = \sqrt{x}$ . The integral becomes

$$\int 2u \sin u du.$$

Now use integration by parts with  $p = 2u$ ,  $dq = \sin u du$ . The resulting antiderivative is

$$-2u\cos u + 2\sin u + C = 2(\sin \sqrt{x} - \sqrt{x} \cos \sqrt{x}) + C.$$

61. Use integration by parts with

$$u = \arcsin x \quad \text{and} \quad dv = \frac{1}{x^2} dx.$$

Then apply Formula 60 of the endpapers, or use the trigonometric substitution  $x = \sin u$ .

62. Let  $x = 3 \sec u$  to transform the integral into

$$9 \int \sec u \tan^2 u du = 9 \int (\sec^3 u - \sec u) du.$$

Apply Formulas 14 and 28 of the endpapers to obtain

$$\begin{aligned} & \frac{9}{2} (\sec u \tan u - \ln|\sec u + \tan u|) + C_1 \\ &= \frac{9}{2} \left( \frac{1}{9} x(x^2 - 9)^{1/2} - \ln \left| \frac{x}{3} \right| + \frac{1}{3} (x^2 - 9)^{1/2} \right) + C_1 \\ &= \frac{1}{2} x(x^2 - 9)^{1/2} - \frac{9}{2} \ln|x| + (x^2 - 9)^{1/2} + C. \end{aligned}$$

63. Let  $x = \sin u$  to transform the integrand into

$$\frac{1}{4} (2 \sin u \cos u)^2 = \frac{1}{4} \sin^2(2u) = \frac{1}{8} (1 - \cos 4u).$$

64. Because  $2x - x^2 = 1 - (x - 1)^2$ , let  $x = 1 + \sin u$ . Then

$$\int x(2x - x^2)^{1/2} dx = \int (1 + \sin u)(\cos^2 u) du$$

$$= \int \left( \frac{1 + \cos 2u}{2} + \cos^2 u \sin u \right) du$$

$$\begin{aligned}
 &= \frac{1}{2} u + \frac{1}{2} \sin u \cos u - \frac{1}{3} \cos^3 u + C \\
 &= \frac{1}{2} \sin^{-1}(x-1) + \frac{1}{2} (x-1)(2x-x^2)^{1/2} \\
 &\quad - \frac{1}{3} (2x-x^2)^{3/2} + C.
 \end{aligned}$$

The answer can be further simplified to

$$\frac{1}{2} \sin^{-1}(x-1) + \frac{1}{6} (2x-x^2)^{1/2} (2x^2-x-3) + C.$$

65. Write

$$\frac{x-2}{(2x+1)^2} = \frac{2x+1}{2(2x+1)^2} - \frac{5}{2(2x+1)^2}.$$

66. Because

$$\frac{2x^2 - 5x - 1}{x^3 - 2x^2 - x + 2} = \frac{1}{x+1} + \frac{2}{x-1} - \frac{1}{x-2},$$

the required antiderivative can be written as

$$\ln \left| \frac{(x+1)(x-1)^2}{x-2} \right| + C.$$

68. Let  $u = \sin x$ . Then  $du = \cos x dx$ , and we obtain

$$\begin{aligned}
 \int \frac{1}{u^2 - 3u + 2} du &= \int \frac{1}{(u-1)(u-2)} du \\
 &= \int \left( -\frac{1}{u-1} + \frac{1}{u-2} \right) du \\
 &= \ln \left| \frac{u-2}{u-1} \right| + C = \ln \left( \frac{2-\sin x}{1-\sin x} \right) + C.
 \end{aligned}$$

69. The partial fractions decomposition of the integrand is

$$\frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{5}{(x+1)^4}.$$

70. The substitution  $u = \tan x$  yields

$$\begin{aligned}\int \frac{1}{u^2 + 2x + 2} du &= \int \frac{1}{1 + (u+1)^2} du \\ &= \tan^{-1}(u+1) + C = \tan^{-1}(1 + \tan x) + C.\end{aligned}$$

71. The partial fractions decomposition of the integrand is

$$\frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}.$$

72. The substitution  $u = \tan \frac{\xi}{2}$  transforms the integral into

$$\int \frac{8 + 4u^2}{(3u^2 + 1)(u^2 + 1)} du.$$

The partial fractions decomposition of the integrand is

$$\frac{10}{3u^2 + 1} - \frac{2}{u^2 + 1},$$

and this leads to the antiderivative

$$\begin{aligned}&\frac{10}{3} \sqrt{3} \tan^{-1}(u\sqrt{3}) - 2 \tan^{-1} u + C \\ &= \frac{10}{3} \sqrt{3} \tan^{-1}(\sqrt{3} \tan \frac{\xi}{2}) - \xi + C.\end{aligned}$$

73. Let  $u = x^3 - 1$ ; the rest is routine.

74. Let  $u = \tan \frac{\xi}{2}$ . Then the integral becomes

$$\begin{aligned}\int \frac{1}{u+2} du &= \ln|u+2| + C \\ &= \ln|2 + \tan \frac{\xi}{2}| + C \\ &= \ln \left| \frac{2 + 2 \cos \xi + \sin \xi}{1 + \cos \xi} \right| + C.\end{aligned}$$

76. Let  $x = \tan^3 z$ . The integral becomes

$$\int \frac{3 \tan^2 z \sec^2 z}{\sec^2 z \tan^2 z} dz = 3z + C = 3 \tan^{-1}(x^{1/3}) + C.$$

77. Use the identity  $\sin 2x = 2 \sin x \cos x$ .

78. Because  $\frac{1}{2}(1 + \cos t) = \cos^2(\frac{t}{2})$ ,

$$\begin{aligned}\int (1 + \cos t)^{1/2} dt &= \sqrt{2} \int (\frac{1 + \cos t}{2})^{1/2} dt \\ &= \sqrt{2} \int \cos(\frac{t}{2}) dt = 2\sqrt{2} \sin(\frac{t}{2}) + C,\end{aligned}$$

which may also be written in the form  $2\sqrt{1 - \cos t} + C$ .

Note: We took the positive square root in the computations above. If this problem had been a definite integral, we'd need to see whether the values of  $t$  made  $\cos(t/2)$  positive or negative to know which sign to take.

79. Multiply numerator and denominator of the integrand by

$$\sqrt{1 - \sin t}.$$

80. Let  $u = \tan t$ . Then the integral becomes

$$\int \frac{1}{1 - u^2} du = \frac{1}{2} \int \left( \frac{1}{1 - u} + \frac{1}{1 + u} \right) du$$

$$= \frac{1}{2} \ln \left| \frac{1 + u}{1 - u} \right| + C = \frac{1}{2} \ln \left| \frac{1 + \tan t}{1 - \tan t} \right| + C.$$

81. Use integration by parts with  $u = \ln(x^2 + x + 1)$  and -- if you wish --  $v = x + \frac{1}{2}$  (this will save some trouble later).

82. Let  $u = e^x$ . The integral becomes

$$I = \int \sin^{-1} u du.$$

Now do an integration by parts with  $p = \sin^{-1} u$ ,  $dq = du$ :

$$I = u \sin^{-1} u - \int u (1 - u^2)^{-1/2} du$$

$$= u \sin^{-1} u + (1 - u^2)^{1/2} + C$$

$$= e^x \sin^{-1}(e^x) + (1 - e^{2x})^{1/2} + C.$$

83. Integration by parts, with  $u = \arctan x$ ,  $dv = x^{-2} dx$ . The new integrand has the partial fractions decomposition

$$\frac{1}{x} - \frac{x}{x^2 + 1}.$$

84. Let  $x = 5 \sec u$ . The integral is transformed into

$$25 \int \sec^3 u \, du;$$

application of Formula 28 from the endpapers and subsequent resubstitution yields the answer

$$\frac{1}{2} \{x(x^2 - 25)^{1/2} + 25 \ln|x| + (x^2 - 25)^{1/2}\} + C$$

(an extra constant in the logarithmic term has been absorbed by the constant  $C$  of integration).

85. The partial fractions decomposition of the integrand is

$$\frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2},$$

and integration of these terms presents no difficulties.

86. Because  $6x - x^2 = 9 - (x - 3)^2$ , we use the substitution  $x = 3 + 3 \sin u$ . This transforms the integral into

$$\begin{aligned} \frac{1}{3} \int \frac{1}{1 + \sin u} \, du &= \frac{1}{3} \int \frac{1 - \sin u}{\cos^2 u} \, du \\ &= \frac{1}{3} \int (\sec^2 u - \frac{\sin u}{\cos^2 u}) \, du = \frac{1}{3} (\tan u - \frac{1}{\cos u}) + C \\ &= \frac{-1 + \sin u}{3 \cos u} + C = \frac{x - 6}{3(6x - x^2)^{1/2}} + C. \end{aligned}$$

87. Let  $x = 2 \tan \xi$ ; this substitution transforms the integral into

$$\int (\frac{1}{2} \cos u + \frac{3}{2} \sin u) \, du,$$

and the rest is routine.

88. Use integration by parts with  $u = \ln x$ ,  $dv = x^{3/2} dx$ . The antiderivative is

$$\frac{2}{25}x^{5/2}(-2 + 5\ln x) + C.$$

89. If  $u = 1 + \sin^2 x$ , then the integral becomes

$$\frac{1}{2} \int \sqrt{u} du,$$

and there are no further difficulties.

90. Let  $u = \sqrt{\sin x}$ . The integral becomes

$$\int 2e^u du = 2e^u + C = 2 \exp(\sqrt{\sin x}) + C.$$

91. By parts: A good choice is to let  $u = x$  and  $dv = e^x \sin x dx$ . It turns out that one must antiderivative both  $e^x \sin x$  and  $e^x \cos x$ , but here Formulas 67 and 68 of the endpapers may be used, or integration by parts will suffice for each.

92. By parts: Let  $u = x^{3/2}$ ,  $dv = x^{1/2} \exp(x^{3/2}) dx$ . The antiderivative is

$$\frac{2}{3}(x^{3/2} - 1)\exp(x^{3/2}) + C.$$

93. Let  $u = \arctan x$ ,  $dv = (x - 1)^{-3} dx$ . Next, after the integration by parts, one confronts the integral

$$\frac{1}{2} \int \frac{1}{(1 + x^2)(x - 1)^2} dx.$$

The partial fractions decomposition of the integrand is

$$\frac{1}{2} \left( \frac{x}{1+x^2} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right).$$

The rest is routine.

94. Use integration by parts, with  $u = \ln(1 + \sqrt{x})$  and  $dv = dx$ . If you choose  $v = x - 1$ , certain difficulties are skirted, and the resulting antiderivative is

$$(x-1)\ln(1+\sqrt{x}) - \frac{1}{2}x + \sqrt{x} + C.$$

95. Because  $3 + 6x - 9x^2 = 4 - (3x-1)^2$ , we use the substitution

$$x = \frac{1}{3}(1 + 2\sin u).$$

The integral is thereby transformed into

$$\frac{1}{9} \int (11 + 4\sin u) du,$$

and the rest is standard.

96. Let  $u = \tan \frac{\xi}{2}$ . The integral becomes

$$\begin{aligned} \int \frac{2}{u^2 + 4u + 3} du &= \int \left( \frac{1}{u+1} - \frac{1}{u+3} \right) du \\ &= \ln \left| \frac{u+1}{u+3} \right| + C = \ln \left| \frac{\frac{1}{3} + \tan(\xi/2)}{3 + \tan(\xi/2)} \right| + C. \end{aligned}$$

The transformations shown in the solution of Problem 16 of Section 9.8 allow you to write this answer in the form

$$\ln \left| \frac{1 + \sin \xi + \cos \xi}{3 + \sin \xi + 3 \cos \xi} \right| + C.$$

97. Multiply numerator and denominator by  $\cos \xi + 1$ . Because

$\cos^2 \xi - 1 = -\sin^2 \xi$ , the integral becomes

$$\int (-\sin \xi \cos \xi - \sin \xi) d\xi \\ = -\frac{1}{2} \cos^2 \xi + \cos \xi + C.$$

98. Use integration by parts with  $u = \tan^{-1} \sqrt{x}$  and  
 $dv = x^{3/2} dx$ . Result:

$$\frac{2}{5} x^{5/2} \tan^{-1} \sqrt{x} - \frac{1}{5} \int \frac{x^2 - 1 + 1}{x + 1} dx \\ = \frac{2}{5} x^{5/2} \tan^{-1} \sqrt{x} - \frac{1}{10} x^2 + \frac{1}{5} x - \frac{1}{5} \ln|x + 1| + C.$$

99. Use integration by parts with  $u = \sec^{-1} \sqrt{x}$ ,  $dv = dx$ .

100. Let  $u = x^2$ . The integral becomes

$$\frac{1}{2} \int \left( \frac{1-u}{1+u} \right)^{1/2} du.$$

Now let  $v^2 = \frac{1-u}{1+u}$ ; then  $u = \frac{1-v^2}{1+v^2}$  and

$$du = -\frac{4v}{(1+v^2)^2} dv.$$

The integral is thereby converted into

$$-\frac{1}{2} \int \frac{4v^2}{(1+v^2)^2} dv.$$

Finally let  $v = \tan z$ . Then we obtain

$$-2 \int \sin^2 z \, dz = \sin z \cos z - z + C.$$

Subsequent resubstitutions and simplifications lead to the answer:

$$\frac{1}{2} (1 - x^4)^{1/2} - \tan^{-1}(\{1 - x^2\}/\{1 + x^2\})^{1/2} + C.$$

101. The area is

$$\begin{aligned} A &= \int_0^1 2\pi \cosh^2 x \, dx \\ &= 2\pi \left( \frac{1}{4} \sinh 2 + \frac{1}{2} - \frac{1}{4} \sinh 0 - 0 \right) \\ &= \frac{\pi}{4} (e^2 - \frac{1}{e^2} + 4) \sim 8.83865. \end{aligned}$$

102. The length of the curve is

$$L = \int_0^1 (1 + e^{-2x})^{1/2} \, dx.$$

Now let  $e^{-x} = \tan u$ . Then

$$\begin{aligned} L &= \int_{x=0}^1 -\frac{\sec^3 u}{\tan u} \, du \\ &= - \int_{x=0}^1 (\csc u + \sec u \tan u) \, du \\ &= \left[ -\ln|\csc u - \cot u| - \sec u \right]_{x=0}^1 \end{aligned}$$

$$\begin{aligned}
&= \left[ -\ln \left| \frac{(1 + e^{-2x})^{1/2} - 1}{e^{-x}} \right| - (1 + e^{-2x})^{1/2} \right]_0^1 \\
&= \sqrt{2} + \ln(\sqrt{2} - 1) - \ln(\{e^2 + 1\}^{1/2} - e) - \frac{1}{e}(e^2 + 1)^{1/2}
\end{aligned}$$

-- which is approximately equal to 1.1927014.

$$103. A_b = \int_0^b 2\pi e^{-x} (1 + e^{-2x})^{1/2} dx.$$

Use the substitution  $u = e^{-x}$ . It follows that

$$\begin{aligned}
A_b &= \int_1^{e^{-b}} 2\pi u (1 + u^2)^{1/2} \left( -\frac{1}{u} \right) du \\
&= \int_p^1 2\pi (1 + u^2)^{1/2} du \quad (\text{where } p = e^{-b}).
\end{aligned}$$

Let  $u = \tan z$ . Then

$$\begin{aligned}
A_b &= \int_{u=p}^1 2\pi (1 + \tan^2 z)^{1/2} \sec^2 z dz \\
&= \pi \left[ \sec z \tan z + \ln |\sec z + \tan z| \right]_{u=p}^1 \\
&= \pi \{ \sqrt{2} + \ln(1 + \sqrt{2}) - e^{-b} (1 + e^{-2b})^{1/2} \\
&\quad - \ln |e^{-b} + (1 + e^{-2b})^{1/2}| \}.
\end{aligned}$$

The limit as  $b \rightarrow \infty$  is

$$A_\infty = \pi \{ \sqrt{2} + \ln(1 + \sqrt{2}) \} \sim 7.2118.$$

$$104. A_b = \int_1^b \frac{2\pi}{x^3} (x^4 + 1)^{1/2} dx.$$

Let  $x^2 = \tan u$ . Then

$$\begin{aligned} A_b &= \pi \int_{x=1}^b \frac{\sec u}{\tan^2 u} \sec^2 u du \\ &= \pi \int_{x=1}^b \left( \frac{\cos u}{\sin^2 u} + \sec u \right) du \\ &= \pi \left[ -\csc u + \ln|\sec u + \tan u| \right]_{x=1}^b \\ &= \pi \left[ -\frac{(x^4 + 1)^{1/2}}{x^2} + \ln\{x^2 + (x^4 + 1)^{1/2}\} \right]_{x=1}^b \\ &= \pi \left( \ln\left(\frac{b^2 + (b^4 + 1)^{1/2}}{1 + 2^{1/2}}\right) + \sqrt{2} - \frac{(b^4 + 1)^{1/2}}{b^2} \right). \end{aligned}$$

Next,  $A_b \rightarrow +\infty$  as  $b \rightarrow +\infty$ , because for large  $b$ ,

$$\frac{(b^4 + 1)^{1/2}}{b^2} \sim 1$$

and

$$\ln\{b^2 + (b^4 + 1)^{1/2}\} \sim \ln(2b^2).$$

105. Because

$$1 + \left(\frac{dy}{dx}\right)^{1/2} = \frac{2x^2 - 1}{x^2 - 1},$$

the surface area is

$$A = \int_1^2 2\pi(x^2 - 1)^{1/2} \left( \frac{2x^2 - 1}{x^2 - 1} \right)^{1/2} dx,$$

$$= 2\pi \int_1^2 (2x^2 - 1)^{1/2} dx.$$

The substitution  $x = 2^{-1/2} \sec u$  leads to the antiderivative

$$2^{-1/2} \pi \{\sec u \tan u - \ln|\sec u + \tan u|\} + C,$$

and thereby to the answer given in the text.

106. Part (a): Use integration by parts with  $u = (\ln x)^n$  and  $dv = x^m dx$ .

Part (b): Application of the formula and simplification of the result yields

$$\frac{17e^4 + 3}{128} \sim 7.2747543.$$

108. Answer:  $\frac{8}{693} \sim 0.011544$ .

109. The area is

$$A = 2 \int_0^2 x^{5/2} (2 - x)^{1/2} dx.$$

The suggested substitution  $x = 2 \sin^2 \xi$  yields

$$A = 64 \int_0^{\pi/2} \sin^6 \xi \cos^2 \xi d\xi.$$

Then the formula of Problem 108 gives the answer  $\frac{5}{4}\pi$ .

110. After division, the integral becomes

$$\int_0^1 (t^6 - 4t^5 + 5t^4 - 4t^2 + 4 - \frac{4}{t^2 + 1}) dt,$$

and its value is  $\frac{22}{7} - \pi$ . Consequently  $\pi < \frac{22}{7}$  (why?).

112. Because  $\{1 + (dy/dx)^2\}^{1/2} = (1 + \sqrt{x})^{1/2}$ , the length of the curve is

$$L = \int_0^1 (1 + \sqrt{x})^{1/2} dx.$$

Let  $x = \tan^4 u$ . We obtain

$$L = \int_0^{\pi/4} 4(1 + \tan^2 u)^{1/2} \tan^3 u \sec^2 u du$$

$$= 4 \int_0^{\pi/4} \sec^3 u \tan^3 u du$$

$$= 4 \int_0^{\pi/4} (\sec^3 u)(\sec^2 u - 1) \tan u du$$

$$= 4 \int_0^{\pi/4} (\sec^4 u - \sec^2 u)(\sec u \tan u) du$$

$$= \left[ \frac{4}{5} \sec^5 u - \frac{4}{3} \sec^3 u \right]_0^{\pi/4}$$

$$= \frac{4}{5} (4\sqrt{2} - 1) - \frac{4}{3} (2\sqrt{2} - 1)$$

$$= \frac{8}{15} (1 + \sqrt{2}) \sim 1.28758.$$

113. Its length is

$$\begin{aligned} L &= \int_1^4 (1 + x^{-1/2})^{1/2} dx \\ &= \int_1^4 \frac{(1 + x^{1/2})^{1/2}}{x^{1/4}} dx. \end{aligned}$$

Now let  $x = u^4$ . We find that

$$\begin{aligned} L &= \int_1^{\sqrt{2}} \frac{(1 + u^2)^{1/2}}{u} 4u^3 du \\ &= 4 \int_1^{\sqrt{2}} u^2 (1 + u^2)^{1/2} du. \end{aligned}$$

Next, let  $u = \tan z$ , and write \* for  $\pi/4$  and \*\* for  $\arctan(\sqrt{2})$ . Then

$$\begin{aligned} L &= 4 \int_*^{**} (\sec^5 z - \sec^3 z) dz \\ &= \left[ \sec^3 z \tan z - \left( \frac{1}{2} \sec z \tan z + \frac{1}{2} \ln |\sec z + \tan z| \right) \right]_*^{**} \\ &= \frac{5}{2} \sqrt{6} - \frac{3}{2} \sqrt{2} + \frac{1}{2} \ln \left( \frac{1 + 2^{1/2}}{3^{1/2} + 2^{1/2}} \right) \sim 3.869983. \end{aligned}$$

114. Let  $Q(t)$  denote the amount of water (in cubic feet) in the tank at time  $t \geq 0$ , and  $y(t)$  the depth of water then. Because the radius then is  $\frac{1}{2}y(t)$ , it follows that

$$Q(t) = \frac{\pi \{y(t)\}^3}{12}.$$

We know also that  $Q(0) = 0$ , and by Torricelli's law that

$$\frac{dQ}{dt} = 50 - 10\sqrt{y(t)}.$$

But

$$\frac{dQ}{dt} = \frac{dQ}{dy} \cdot \frac{dy}{dt} = \frac{3\pi}{12} \{y(t)\}^2 \frac{dy}{dt},$$

and therefore

$$\frac{dy}{dt} = \frac{50 - 10y^{1/2}}{(\pi/4)y^2}.$$

Let  $T$  denote the time required to fill the tank. Then

$$\begin{aligned} T &= \int_0^T 1 dt = \int_{t=0}^T \frac{(\pi/4)y^2}{50 - 10y^{1/2}} dy \\ &= \int_0^9 \frac{(\pi/4)y^2}{10(5 - y^{1/2})} dy = \frac{\pi}{40} \int_0^9 \frac{y^2}{5 - y^{1/2}} dy. \end{aligned}$$

The substitution  $y = x^2$  yields

$$\begin{aligned} T &= \frac{\pi}{40} \int_0^3 \frac{2x^5}{5 - x} dx \\ &= -\frac{\pi}{20} \int_0^3 (x^4 + 5x^3 + 25x^2 \\ &\quad + 125x + 625 + \frac{3125}{x-5}) dx \end{aligned}$$

$$= \frac{\pi}{400} \{62500 \ln(5/2) - 56247\} \sim 8.020256 \text{ (seconds).}$$

115. Use the substitution  $u = e^x$ . The integral then becomes

$$\int \frac{1}{u^2 + u + 1} du,$$

and we use now-familiar techniques to obtain the answer.

116. Part (a): The iteration

$$x \leftarrow x - \frac{x^3 + x + 1}{3x^2 + 1} = \frac{2x^3 - 1}{3x^2 + 1}$$

with initial value  $x_0 = -0.5$  yields the approximate root  
 $-0.682327804$ .

Part (b): If we let  $r$  denote the root obtained in Part (a), division yields the quotient  $x^2 + rx + (r^2 + 1)$ ; that is, the irreducible quadratic factor is

$$x^2 - (0.6823278)x + 1.4655712$$

(coefficients approximate, of course).

Part (c): Let  $q = r^2 + 1$ . The partial fractions decomposition of the integrand is

$$\frac{A}{x - r} + \frac{Bx + C}{x^2 + rx + q}$$

where

$$A = \frac{1}{3r^2 + 1}, \quad B = -\frac{1}{3r^2 + 1}, \quad \text{and}$$

$$C = -\frac{2r}{3r^2 + 1}.$$

This yields the antiderivative

$$\frac{1}{3r^2 + 1} \left[ \ln \left| \frac{r - 1}{r} \right| + \frac{1}{2} \ln \left| \frac{q}{1 + r + q} \right| \right]$$

(continued)

$$+ \left[ -\frac{3r}{2w} \left\{ \tan^{-1} \left( \frac{r}{2w} \right) - \tan^{-1} \left( \frac{r+2}{2w} \right) \right\} \right] + C$$

$$\text{where } w = \frac{1}{2} (3r^2 + 4)^{1/2}.$$

With  $r = -0.6823279038$ , we find that  $q \sim 1.465571232$ ,  $w \sim 1.16154140$ , and thereby that

$$\int_0^1 \frac{1}{x^3 + x + 1} dx \sim 0.6303193226.$$

117. First multiply numerator and denominator by  $e^{-x}$ .

118. With the recommended substitution, the numerator is  $\frac{1}{2} du$ , so the integral becomes

$$\begin{aligned} \frac{1}{2} \int u^{-3} du &= -\frac{1}{4} u^{-2} + C \\ &= -\frac{1}{4(x^4 + x^2)^2} + C. \end{aligned}$$

119. You should obtain

$$\begin{aligned} \int \sqrt{\tan \xi} d\xi &= \int \frac{u^{1/2}}{1+u^2} du \\ &= \int \frac{2x^2}{1+x^4} dx. \end{aligned}$$