For this quiz, we have:

$$y = e^x \text{ and the center } x_0 = 0.$$

(a). (2 pts) Find a general form for the n^{th} Taylor Coefficient. **Your answer can have an n in it, but not any x's or f's.**

$$c_n = \frac{1}{n!}$$

Solution: The n^{th} Taylor Coefficient of $y = f(x)$ at the center x_0 is $c_n := \frac{f^{(n)}(x_0)}{n!}$. Here, $f(x) = e^x$. We do not even have to make a chart to see the pattern for the n^{th}-derivative of f since it is clear that $f^{(n)}(x) = e^x$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. As the center $x_0 = 0$, we have $f^{(n)}(x_0) = e^0 = 1$ for every integer $n \geq 0$. So $c_n = \frac{f^{(n)}(x_0)}{n!} = \frac{1}{n!}$.

(b). (4 pts) Find the Taylor Series of $y = e^x$ about the center $x_0 = 0$.

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \text{Also fine: } P_\infty(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Solution: Recall that the general formula for a Taylor Series $y = P_\infty(x)$ of f about the center x_0 is:

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

In part (a), we computed $\frac{f^{(n)}(x_0)}{n!} = \frac{1}{n!}$ in part (a). As $x_0 = 0$, $(x - x_0)^n = (x - 0)^n = x^n$.

The correct answer for $P_\infty(x)$ is in the box above.

(c). (4 pts) Using Taylor's Remainder Theorem (i.e., the Big Theorem from the class handout), show that the power series you found in part (b) converges for $x \in (-2, 2)$.

Solution: Our goal is to carefully show that $e^x = P_\infty(x)$ for each $x \in (-2, 2)$. Recall that

$$e^x = P_N(x) + R_N(x) \tag{1}$$

where P_N is the N^{th}-order Taylor Polyonimal and R_N is the N^{th}-order Taylor Remainder. The remainder/error term R_N is defined so that (1) holds, i.e., R_N is defined as $R_N(x) := e^x - P_N(x)$. So to show that

$$\text{for each } x \in (-2, 2), \qquad e^x = P_\infty(x) \ ,$$

we need to show that

$$\text{for each } x \in (-2, 2), \qquad \lim_{N\to\infty} |R_N(x)| = 0 \ . \tag{2}$$

- **Step 1:** Find a good upper bound for $|R_n(x)|$ for $x \in (-2,2)$.
Consider an $x \in (-2,2)$. By Taylor's Remainder Theorem

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

for some c between x and the center 0. Since $x \in (-2,2)$ and c is between x and 0, we know that $c \in (-2,2)$. Thus

$$|R_N(x)| \underset{\substack{\uparrow \\ \text{Taylor Remainder Formula}}}{=} \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right| \overset{\text{Ⓐ}}{=} \frac{\left|f^{(N+1)}(c)\right|}{(N+1)!} |x|^{N+1} \underset{\substack{\uparrow \\ \text{by part (a)}}}{=} \frac{e^c}{(N+1)!} |x|^{N+1} \underset{\substack{\uparrow \\ \text{since } x \in (-2,2),\ c \in (-2,2)}}{\leq} \frac{e^2}{(N+1)!} 2^{N+1}.$$

So a good upper bound for $|R_N(x)|$, for $x \in (-2,2)$, is

$$|R_N(x)| \leq e^2 \frac{2^{N+1}}{(N+1)!}. \tag{3}$$

- **Step 2:** Show that the (good) upper bound we found in (3) tends to zero as $N \to \infty$.
So we want to show that $\lim_{N\to\infty} \frac{e^2\, 2^{N+1}}{(N+1)!} = 0$. (Sometimes this step is easy but in this example we will have to use a little trick (tool) that sometimes works ... here we go). Let $a_N = \frac{e^2\, 2^{N+1}}{(N+1)!}$. (To show that $\lim_N a_N = 0$, we will actually show something stronger, namely $\sum a_N$ converges.) The Ratio Test tells us that the series

$$\sum_{n=0}^{\infty} \frac{e^2}{(n+1)!} 2^{n+1}$$

is (absolutely) convergent since applying the Ratio Test we get

$$\rho = \lim_{N\to\infty} \left| \frac{a_{N+1}}{a_N} \right| = \lim_{N\to\infty} \frac{e^2\, 2^{N+2}}{(N+2)!} \cdot \frac{(N+1)!}{e^2\, 2^{N+1}} = \lim_{N\to\infty} \frac{2}{N+2} = 0.$$

The n^{th} term test for divergence gives that if the series $\sum_n a_n$ converges, then the limit of the sequence $\{a_n\}_n$ is 0, i.e. $\lim_{n\to\infty} a_n = 0$. So $\lim_{N\to\infty} \frac{e^2\, 2^{N+1}}{(N+1)!} = 0$.
- **Step 3:** Show that $\lim\limits_{N\to\infty} |R_N(x)| = 0$.

$$0 \leq |R_N(x)| \overset{\text{by (3)}}{\leq} \frac{2^{N+1}}{(N+1)!} e^2 \xrightarrow{\text{as } n\to\infty, \text{ by Step 2}} 0.$$

The Squeeze/Sandwich Theorem gives that $\lim_{N\to\infty} |R_N(x)| = 0$.