Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

a. The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

b. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$$P_N(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

c. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n$$

d. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots$$

e. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n$$

f. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}\,(x-x_0)^{(N+1)}$$

for some c between x and x_0.

g. A Maclaurin series is a Taylor series with the center specifically specified as $x_0 = 0$.