Math 142 **§ 10.7 Taylor and Maclaurin Polynominals**

Read this handout thoroughly and then do Homeworks: 2, 4, 5, and 6 (on a seperate sheet of paper).

Let's consider a function

$$y = f(x)$$

and fix a point x_0 in the domain of $y = f(x)$. So the graph of $y = f(x)$ goes through the point

$$(x_0, f(x_0)) \ .$$

The equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is found by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(x_0) &= f'(x_0)(x - x_0) \\ y &= f(x_0) + f'(x_0)(x - x_0) \end{aligned}$$

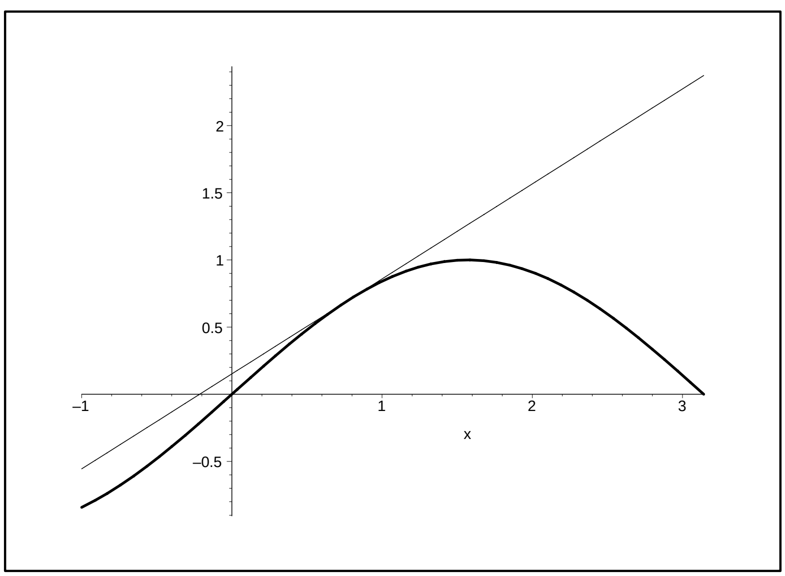
So the equation $y = p_1(x)$ of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0) \ . \tag{1}$$

Recall that the function $y = f(x)$ can be approximated *locally* near x_0 by this tangent line $y = p_1(x)$. In other words, if x is close to x_0 then the value $f(x)$ is close to the value $p_1(x)$, that is, if $x \approx x_0$ then $f(x) \approx p_1(x)$. (Draw yourself a picture.)

Example 1. $f(x) = \sin(x)$ near $x_0 = \frac{\pi}{4}$ can be approximated by the line

$$\begin{aligned} y &= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) \\ y &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) \end{aligned}$$



Homework 2. Find the equation of the tangent line to the function $f(x) = \dfrac{1}{x}$ at the point $x_0 = 2$.

Note that this tangent line approximation works well because the tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$ is *the only line with slope $f'(x_0)$ passing through the point* $(x_0, f(x_0))$. We can generalize this to second degree approximations by finding the parabola passing through the point $(x_0, f(x_0))$ with the same slope (first derivative) as $y = f(x)$ at x_0 and the same *second* derivative as $y = f(x)$ at x_0.

Example 3. Consider $f(x) = e^{-(x-1)}$ at $x_0 = 1$. Such a parabola as we want looks like

$$p_2(x) = c_0 + c_1(x-1) + c_2(x-1)^2$$

and we can find the **constants** c_0, c_1, c_2 to make the derivatives of $y = f(x)$ and $y = p_2(x)$ match up at $x_0 = 1$. We have

$$\begin{aligned} p_2(x) &= c_0 + c_1(x-1) + c_2(x-1)^2 && \text{and} && f(x) = e^{-(x-1)} \\ p_2'(x) &= c_1 + 2c_2(x-1) && \text{and} && f'(x) = -e^{-(x-1)} \\ p_2''(x) &= 2c_2 && \text{and} && f''(x) = +e^{-(x-1)} \end{aligned}$$

and evaluating these at $x_0 = 1$ gives us

$$\begin{aligned} p_2(1) &= c_0 && \text{and} && f(1) = e^{-(0)} = 1 \\ p_2'(1) &= c_1 && \text{and} && f'(1) = -e^{-(0)} = -1 \\ p_2''(1) &= 2c_2 && \text{and} && f''(1) = +e^{-(0)} = 1 \end{aligned}$$

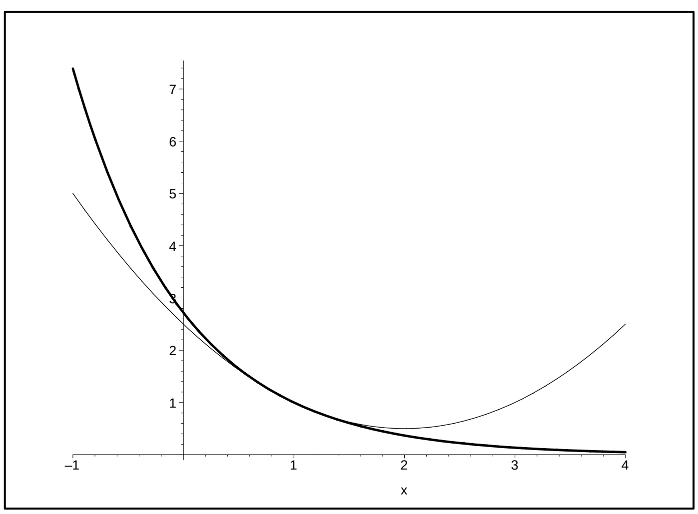
and so

$$\begin{aligned} p_2(1) &= f(1) && \Leftrightarrow && c_0 = 1 \\ p_2'(1) &= f'(1) && \Leftrightarrow && c_1 = -1 \\ p_2''(1) &= f''(1) && \Leftrightarrow && 2c_2 = 1 && \Leftrightarrow && c_2 = \tfrac{1}{2}\ . \end{aligned}$$

So our parabola is

$$p_2(x) = 1 - (x-1) + \frac{1}{2}(x-1)^2.$$

This polynomial is *the second order Taylor polynomial* of $y = e^{-(x-1)}$ centered at $x_0 = 1$. Notice that close to x=1 this parabola approximates the function rather well.



Using this as a model we can give a general form of the second order Taylor polynomial for $y = f(x)$ at x_0, that is the parabola

$$p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2$$

where we want to find the **constants** c_0, c_1, c_2 to make the derivatives of $y = f(x)$ and $y = p_2(x)$ match up at $x = x_0$. We have

$$\begin{array}{lcl} p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 & \Longrightarrow & p_2(x_0) \ = c_0 \\ p_2'(x) = c_1 + 2c_2(x - x_0) & \Longrightarrow & p_2'(x_0) \ = c_1 \\ p_2''(x) = 2c_2 & \Longrightarrow & p_2''(x_0) = 2c_2 \end{array}$$

and so

$$\begin{array}{lclclcl} p_2(x_0) = f(x_0) & \Longleftrightarrow & c_0 = f(x_0) & & \\ p_2'(x_0) = f'(x_0) & \Longleftrightarrow & c_1 = f'(x_0) & & \\ p_2''(x_0) = f''(x_0) & \Longleftrightarrow & 2c_2 = f''(x_0) & \Longleftrightarrow & c_2 = \frac{f''(x_0)}{2} \ . \end{array}$$

So our parablola is

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \ . \qquad (2)$$

Compare the function $y = p_1(x)$ in formula (1) with the function $y = p_2(x)$ in formula (2).
Starting to see a pattern?

Homework 4. Find the second order Taylor polynomial for $f(x) = \frac{1}{x}$ at $x_0 = -2$. Draw a picture.

Higher order Taylor polynomials are found in the same way. For example, the *third order Taylor polynomial* for a function $y = f(x)$ centered at x_0 is

$$p_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \ .$$

In general, the N^{th}*-order Taylor polynomial* for $y = f(x)$ at x_0 is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N \ ,$$

which can also be written as (recall that $0! = 1$)

$$p_N(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N \ . \quad \text{(N - open form)}$$

Formula (N - open form) is in open form. It can also be written in closed form, by using sigma notation, as

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} \ (x - x_0)^n \ . \qquad \text{(N- closed form)}$$

So $y = p_N(x)$ is a polynomial of degree at most N and it has the form

$$p_N(x) \ = \ \sum_{n=0}^{N} c_n \ (x - x_0)^n$$

where the c_n's

$$c_n \ = \ \frac{f^{(n)}(x_0)}{n!}$$

are specially chosen so that

$$\begin{aligned} p_N(x_0) \ &= \ f(x_0) \\ p_N^{(1)}(x_0) \ &= \ f^{(1)}(x_0) \\ p_N^{(2)}(x_0) \ &= \ f^{(2)}(x_0) \\ &\vdots \\ p_N^{(N)}(x_0) \ &= \ f^{(N)}(x_0) \ . \end{aligned}$$

The constant c_n is called the n^{th} Taylor coefficient of $y = f(x)$ about x_0.
The N^{th}*-order Maclaurin polynomial* for $y = f(x)$ is just
the N^{th}-order Taylor polynomial for $y = f(x)$ at $x_0 = 0$ and so it is

$$p_N(x) \ = \ \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \ (x)^n \ .$$

Homework 5. Compute the fifth order Maclaurin polynomial (i.e. $y = p_5(x)$) for $f(x) = \sin(3x)$.

Why are higher order Taylor polynomials better? Let's look at some graphs from Homework 5. To follow are graphs of $y = \sin(3x)$ along with its Maclaurin polynomial $y = p_N(x)$ for $N = 1, 3, 4, 7, 9, 11, 13$.

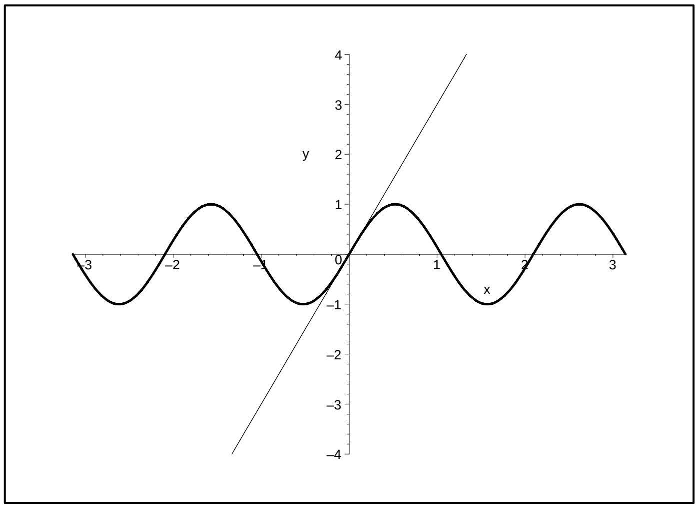


FIGURE 1. $y = \sin(3x)$ along with its first order Maclaurin Polynomial $y = p_1(x)$

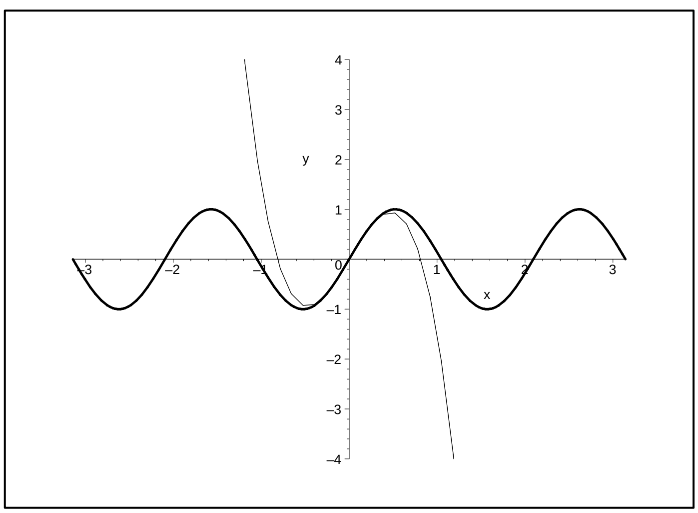


FIGURE 3. $y = \sin(3x)$ along with its third order Maclaurin Polynomial $y = p_3(x)$

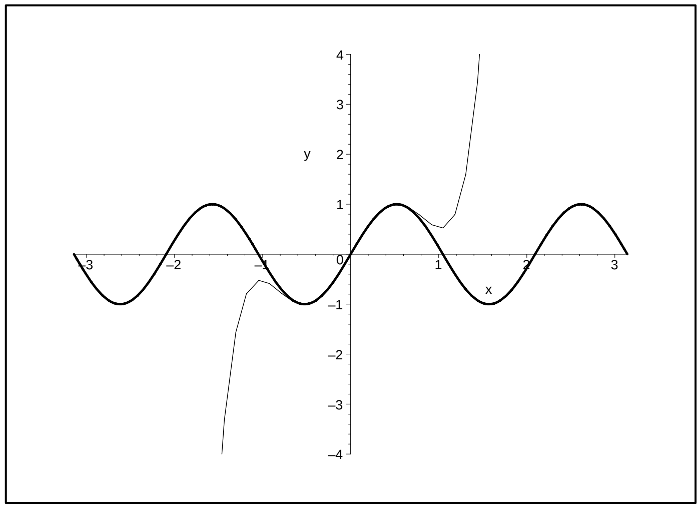


FIGURE 5. $y = \sin(3x)$ along with its fifth order Maclaurin Polynomial $y = p_5(x)$

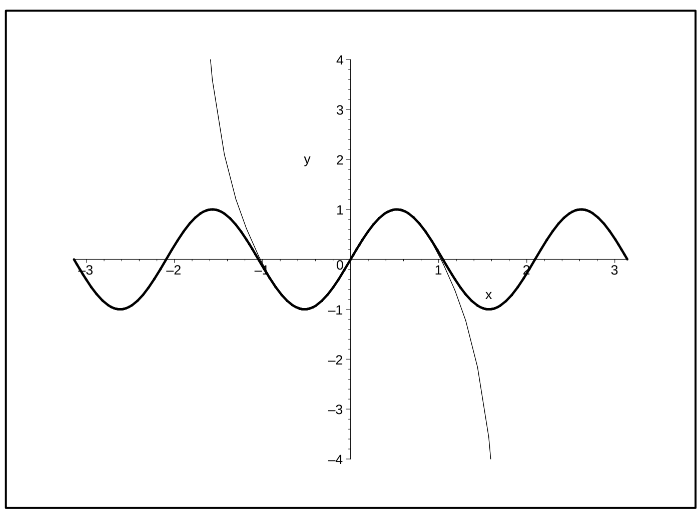


FIGURE 7. $y = \sin(3x)$ along with its 7^{th} order Maclaurin Polynomial $y = p_7(x)$

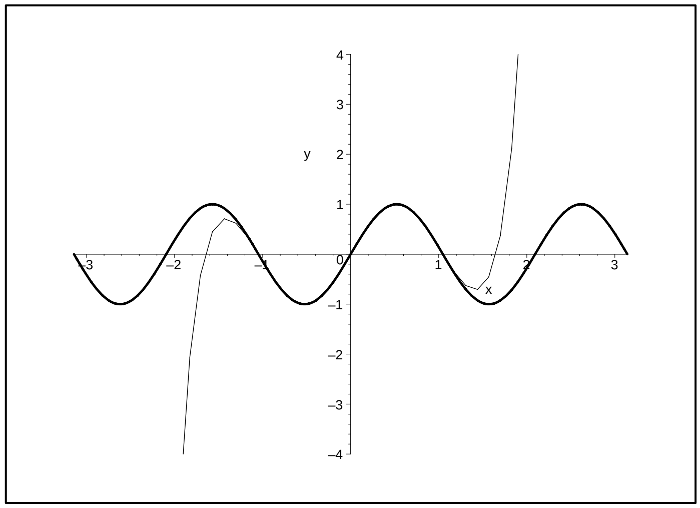


FIGURE 9. $y = \sin(3x)$ along with its 9^{th} order Maclaurin Polynomial $y = p_9(x)$

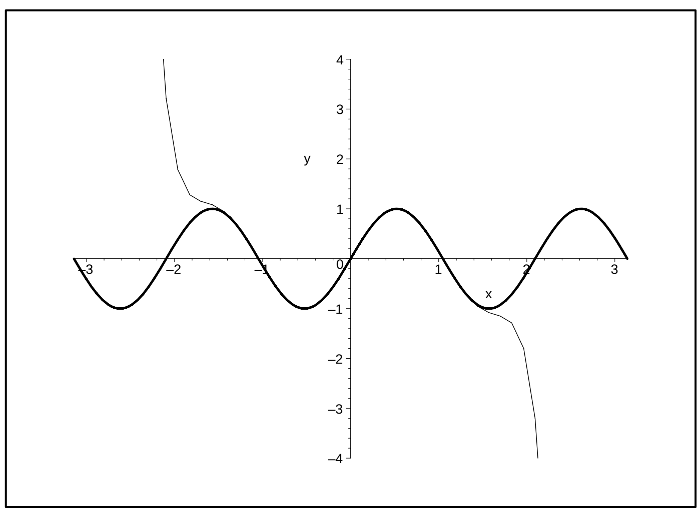


FIGURE 11. $y = \sin(3x)$ along with its 11^{th} order Maclaurin Polynomial $y = p_{11}(x)$

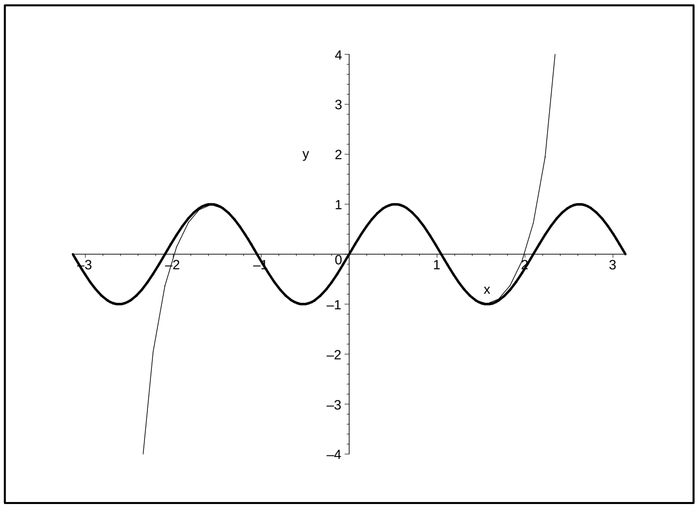


FIGURE 13. $y = \sin(3x)$ along with its 13^{th} order Maclaurin Polynomial $y = p_{13}(x)$

Notice that as N increases the approximation of $y = \sin(3x)$ by $y = p_N(x)$ gets better and better, even over a wider and wider interval around the center $x_0 = 0$. So for a fixed x the approximation of $y = f(x)$ by $y = p_N(x)$ becomes more accurate as N gets bigger.

Homework 6. Find the Maclaurin polynomials: $y = p_1(x)$, $y = p_3(x)$, $y = p_5(x)$, $y = p_7(x)$, $y = p_9(x)$, $y = p_{11}(x)$, and $y = p_{13}(x)$ above (i.e., for $y = \sin(3x)$ about $x_0 = 0$).

Just to think about. Take another look at Homework 6. Do you notice any pattern in the Taylor coefficients? Why did we only use odd-order Taylor polynomials?