Homework on Taylor/Maclaurin Polynomials and Series - **Part 2 # (3)**

Do parts (a) - (i) for the following three problems.

(1) $f(x) = \cos(17x)$ $\quad x_0 = 0$ $\quad J = (-\infty, \infty) = \mathbb{R}$

(2) $f(x) = (1+x)^{-3}$ $\quad x_0 = 0$ $\quad J = \left(0, \frac{1}{2}\right)$

(3) $f(x) = e^x$ $\quad x_0 = \cancel{16}\ 17$ $\quad J = (16, 19)$

You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

Use only:

- the definition of Taylor polynominal
- the definition of Taylor series
- the theorem/error-estimate on the N^{th}-Remainder term for Taylor polynomials.

Do **NOT** use a known Taylor Series (i.e., do not use methods from section 10.10).

3a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) = e^x$	$f^{(0)}(x_0) = e^{17}$
$f^{(1)}(x) = e^x$	$f^{(1)}(x_0) = e^{17}$
$f^{(2)}(x) = e^x$	$f^{(2)}(x_0) = e^{17}$
$f^{(3)}(x) = e^x$	$f^{(3)}(x_0) = e^{17}$
$f^{(4)}(x) = e^x$	$f^{(4)}(x_0) = e^{17}$

3b. Find the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2, 3, 4$.

$P_0(x) = e^{17}$
$P_1(x) = e^{17} + e^{17}(x-17)$
$P_2(x) = e^{17} + e^{17}(x-17) + \frac{e^{17}}{2!}(x-17)^2$
$P_3(x) = e^{17} + e^{17}(x-17) + \frac{e^{17}}{2!}(x-17)^2 + \frac{e^{17}}{3!}(x-17)^3$
$P_4(x) = e^{17} + e^{17}(x-17) + \frac{e^{17}}{2!}(x-17)^2 + \frac{e^{17}}{3!}(x-17)^3 + \frac{e^{17}}{4!}(x-17)^4$

3c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_\infty(x) = e^{17} + e^{17}(x-17) + \frac{e^{17}}{2!}(x-17)^2 + \frac{e^{17}}{3!}(x-17)^3 + \frac{e^{17}}{4!}(x-17)^4 + \ldots$$

3d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{e^{17}}{n!}(x-17)^n$$

3e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

$$c_n = \frac{e^{17}}{n!} \quad \text{for } n = 0, 1, 2, 3, 4, \ldots$$

3f. Find the interval of convergence I of the Taylor series of $y = f(x)$ about x_0. Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$$\boxed{I = (-\infty, \infty)}$$

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{e^{17}}{n!}(x-17)^n$$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e^{17}(x-17)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^{17}(x-17)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \left|\frac{(x-17)^{n+1}}{(x-17)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n!}{n!\,(n+1)} |x-17|$$

$$= |x-17| \lim_{n\to\infty} \frac{1}{n+1} = |x-17| \cdot 0 \overset{?}{<} 1$$

for all x's

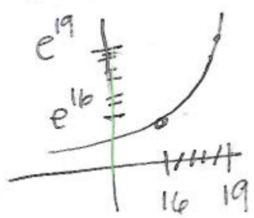
3g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find an upper bound for the maximum of $|f^{(N+1)}(x)|$ on the interval J. You answer can have an N in it but it cannot have an: x, x_0, c. (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\max_{c\in J}\left|f^{(N+1)}(c)\right| \le e^{19}$$

$f^{N+1}(x) = e^x$ for each N so for $c \in J \overset{\text{given}}{=\!=} (16, 19)$

$$|f^{(N+1)}(c)| = |e^c| = e^c \le e^{19}$$

$$\left.\begin{array}{l} 16 < c < 19 \end{array}\right] \Rightarrow e^{16} < e^c < e^{19}$$



3h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find an upper bound for the maximum of $|R_N(x)|$. You answer can have an N and x in it but it cannot have an: x_0, c.

$$|R_N(x)| \le \frac{e^{19}\,|x-17|^{N+1}}{(N+1)!} \quad \underset{\le}{\text{or}} \quad \frac{e^{19}\,2^{N+1}}{(N+1)!}$$

For each $x \in J \overset{\text{given}}{=\!=} (16, 19)$

$$|R_N(x)| \le \max_{c\in J}|f^{(N+1)}(c)| \cdot \frac{1}{(N+1)!}\,|x-17|^{N+1}$$

$$\le e^{19}\,\frac{1}{(N+1)!}\,|x-17|^{N+1}$$

also $x \in J \Rightarrow 16 < x < 19 \Rightarrow -1 < x-17 < 2$

$$\Rightarrow |x-17| < 2$$

3i. Carefully show that $f(x) = P_\infty(x)$ for each x in the given interval J by showing that $\lim_{N\to\infty} |R_N(x)| = 0$ for each $x \in J$.

For each $x \in J$

$$|R_N(x)| \underset{3h}{\leq} e^{19} \frac{2^{N+1}}{(N+1)!} \xrightarrow{N\to\infty} 0 \quad \text{so } f(x) = P_\infty(x)$$

Why? b/c

$$\sum_{N=1}^{\infty} \frac{2^{N+1}}{(N+1)!}$$

(ratio test

$$\rho = \lim_{N\to\infty} \left| \frac{a_{N+1}}{a_N} \right| = \lim_{N\to\infty} \frac{2^{N+2}}{(N+2)!} \frac{(N+1)!}{2^{N+1}} = \lim_{N\to\infty} \frac{2}{N+2} = 0 < 1$$

So by ratio test,

$$\sum_{N=1}^{\infty} \frac{2^{N+1}}{(N+1)!}$$ is (abs.) convergent

so by n^{th} term test

$$\lim_{N\to\infty} \frac{2^{N+1}}{(N+1)!} = 0$$