Homework on Taylor/Maclaurin Polynomials and Series - **Part 2 # (2)**

Do parts (a) - (i) for the following three problems.

(1)	$f(x) = \cos(17x)$	$x_0 = 0$	$J = (-\infty, \infty) = \mathbb{R}$
(2)	$f(x) = (1+x)^{-3}$	$x_0 = 0$	$J = \left(0, \frac{1}{2}\right)$
(3)	$f(x) = e^x$	$x_0 = \cancel{\ } 17$	$J = (16, 19)$

You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

Use only:

- the definition of Taylor polynominal
- the definition of Taylor series
- the theorem/error-estimate on the N^{th}-Remainder term for Taylor polynomials.

Do **NOT** use a known Taylor Series (i.e., do not use methods from section 10.10).

2a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) = (1+x)^{-3} = \frac{+2!}{2}(1+x)^{-3}$	$f^{(0)}(x_0) = 1$
$f^{(1)}(x) = -3(1+x)^{-4} = \frac{-3!}{2}(1+x)^{-4}$	$f^{(1)}(x_0) = -3$
$f^{(2)}(x) = +3.4(1+x)^{-5} = \frac{4!}{2}(1+x)^{-5}$	$f^{(2)}(x_0) = +3.4$
$f^{(3)}(x) = -3.4.5(1+x)^{-6} = \frac{-5!}{2}(1+x)^{-6}$	$f^{(3)}(x_0) = -3.4.5$
$f^{(4)}(x) = +3.4.5.6(1+x)^{-7} = \frac{+6!}{2}(1+x)^{-7}$	$f^{(4)}(x_0) = +3.4.5.6$

2b. Find the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2, 3, 4$.

$P_0(x) = 1$
$P_1(x) = 1 - 3x$
$P_2(x) = 1 - 3x + \frac{3.4}{2!}x^2$
$P_3(x) = 1 - 3x + \frac{3.4}{2!}x^2 - \frac{3.4.5}{3!}x^3$
$P_4(x) = 1 - 3x + \frac{3.4}{2!}x^2 - \frac{3.4.5}{3!}x^3 + \frac{3.4.5.6}{4!}x^4$

$\hookrightarrow$ As for the order in which to do parts (c), (d), (e).
I think it is easiest to do
(#1): (1c) → (1d) → (1e) but for (#2): (2e) → (2d) → (2c).

2c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_\infty(x) = 1 - 3x^1 + \frac{(3)(4)}{2}x^2 - \frac{(4)(5)}{2}x^3 + \frac{(5)(6)}{2}x^4 - \frac{(6)(7)}{2}x^5 + \cdots$$

2d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_\infty(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n$$

2e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

$$c_n = (-1)^n \frac{(n+1)(n+2)}{2}$$

Let make a chart to figure out c_n

n	c_n		
0	1		
1	$-\frac{3}{1!}$	$= \frac{-2\cdot 3}{(2)\ 1!}$	$= -\frac{2\cdot3}{2}$
2	$+\frac{3\cdot4}{2!}$	$= +\frac{2\cdot 3\cdot4}{(2)\ 2!}$	$= \frac{3\cdot4}{2}$
3	$-\frac{3\cdot4\cdot5}{3!}$	$= -\frac{2\cdot3\cdot4\cdot5}{2\ 3!}$	$= -\frac{4\cdot5}{2}$
4	$+\frac{3\cdot4\cdot5\cdot6}{4!}$	$= +\frac{2\cdot3\cdot4\cdot5\cdot6}{2\ 4!}$	$= +\frac{5\cdot6}{2}$
5	$-\frac{3\cdot4\cdot5\cdot6\cdot7}{5!}$	$= \frac{-2\cdot3\cdot4\cdot5\cdot6\cdot7}{(2)\ 5!}$	$= \frac{-6\cdot7}{2}$

$$= (-1)^n \frac{(n+1)(n+2)}{2}$$

also works

$$\frac{(-1)^0 (0+1)(0+2)}{2} = 1$$

$n=0$

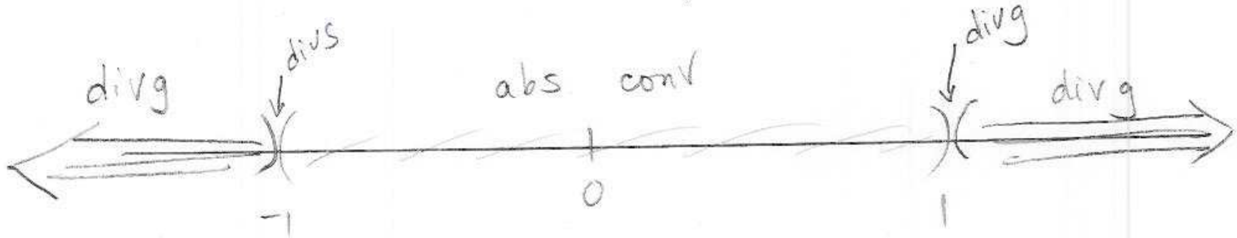
2f. Find the interval of convergence I of the Taylor series of $y = f(x)$ about x_0. Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$I = (-1, 1)$

$$P_\infty(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)(n+3)\, x^{n+1}}{2} \cdot \frac{2}{(n+1)(n+2)\, x^n} \right|$$

$$= |x| \lim_{n\to\infty} \frac{n+3}{n+1} = |x| \cdot 1 \overset{?}{<} 1$$



Check endpoints

$$\underline{x=-1} \quad \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (-1)^n = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \overbrace{\left[(-1)^n (-1)^n\right]}^{=1} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}$$

$$\underline{x=1} \quad \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (1)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2}$$

$$\lim_{n\to\infty} \left| (\pm 1)^n \frac{(n+1)(n+2)}{2} \right| = \frac{1}{2} \lim_{n\to\infty} (n+1)(n+2) = \infty$$

So have divergence at both endpts by n^{th} term test.

$J = (0, 1/2)$

2g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find an upper bound for the maximum of $|f^{(N+1)}(x)|$ on the interval J. You answer can have an N in it but it cannot have an: x, x_0, c. (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\boxed{\max_{c\in J}\left|f^{(N+1)}(c)\right| \le \frac{(N+3)!}{2}}$$

$$(2a) \Rightarrow f^{n}(x) = (-1)^n \frac{(n+2)!}{2}(1+x)^{-(n+3)}. \text{ So if } 0<c<\tfrac{1}{2}$$

$$\Rightarrow |f^{(N+1)}(c)| = \left|(-1)^{N+1}\frac{((N+1)+2)!}{2}(1+c)^{-((N+1)+3)}\right|$$

$$= \frac{(N+3)!}{2}\frac{1}{(1+c)^{N+4}} \underset{4}{\le} \frac{(N+3)!}{2}\frac{1}{(1+0)^{N+4}} = \frac{(N+3)!}{2}$$

$$\boxed{0<c<\tfrac{1}{2} \Rightarrow 1<1+c<\cancel{\tfrac{3}{4}}\ 3/2 \Rightarrow 1^{N+1} < (1+c)^{N+4} < \cancel{\left(\tfrac{3}{4}\right)^{N+4}}\ \left(3/2\right)^{N+4}}$$

2h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find an upper bound for the maximum of $|R_N(x)|$. You answer can have an N and x in it but it cannot have an: x_0, c.

$$\boxed{|R_N(x)| \le \frac{(N+2)(N+3)}{2}|x|^{N+1}}$$

$$|R_N(x)| \le \max_{c\in J}|f^{(N+1)}(c)| \cdot \frac{1}{(N+1)!}|x|^{N+1}$$

$$\le \frac{1}{2}\frac{(N+3)!}{(N+1)!}|x|^{N+1} = \frac{(N+2)(N+3)}{2}|x|^{N+1}$$

2i. Carefully show that $f(x) = P_\infty(x)$ for each x in the given interval J by showing that $\lim_{N\to\infty} |R_N(x)| = 0$ for each $x \in J$.

Let $x \varepsilon J$. So $0 < x < \frac{1}{2}$. So

$$|R_N(x)| \overset{2h}{\leq} \frac{(N+2)(N+3)}{2} \cdot |x|^{N+1}$$

$0 < x < \frac{1}{2}$

$$\leq \frac{(N+2)(N+3)}{2} \left(\frac{1}{2}\right)^{N+2}$$

$$= \frac{1}{2^3} \frac{(N+2)(N+3)}{2^N} \xrightarrow[\text{2-ways}]{N\to\infty} 0$$

WAY # 1 : L'Hopital

$$\lim_{N\to\infty} \frac{(N+2)(N+3)}{2^N} = \lim_{N\to\infty} \frac{N^2+5N+6}{2^N} \underset{L'H}{\overset{\infty/\infty}{=}} \lim_{N\to\infty} \frac{2N+5}{2^N(\ln 2)} \underset{L'H}{\overset{\infty/\infty}{=}}$$

$$= \lim_{N\to\infty} \frac{2}{2^N(\ln 2)^2} = 0.$$

WAY # 2 Helpful Intuition

$$0 \leq \frac{(N+2)(N+3)}{2^N} \underset{\text{N big}}{\leq} \frac{(N+2)(N+3)}{N^3} \xrightarrow{N\to\infty} 0$$

So $\lim_{N\to\infty} |R_N(x)| = 0$