**Homework on Taylor/Maclaurin Polynomials and Series - Part 2 # (1)**

Do parts (a) - (i) for the following three problems.

| | | | |
|---|---|---|---|
| (1) | $f(x) = \cos(17x)$ | $x_0 = 0$ | $J = (-\infty, \infty) = \mathbb{R}$ |
| (2) | $f(x) = (1+x)^{-3}$ | $x_0 = 0$ | $J = \left(0, \frac{1}{2}\right)$ |
| (3) | $f(x) = e^x$ | $x_0 = \cancel{\phantom{0}}\ 17$ | $J = (16, 19)$ |

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You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

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Use only:

- the definition of Taylor polynominal
- the definition of Taylor series
- the theorem/error-estimate on the $N^{\text{th}}$-Remainder term for Taylor polynomials.

Do **NOT** use a known Taylor Series (i.e., do not use methods from section 10.10).

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**1a.** Find the following. Note the first column are functions of $x$ and the second column are numbers.

| | | | |
|---|---|---|---|
| $f^{(0)}(x) =$ | $\cos(17x)$ | $f^{(0)}(x_0) =$ | $1$ |
| $f^{(1)}(x) =$ | $-17 \sin(17x)$ | $f^{(1)}(x_0) =$ | $0$ |
| $f^{(2)}(x) =$ | $-(17)^2 \cos(17x)$ | $f^{(2)}(x_0) =$ | $-(17)^2$ |
| $f^{(3)}(x) =$ | $+(17)^3 \sin(17x)$ | $f^{(3)}(x_0) =$ | $0$ |
| $f^{(4)}(x) =$ | $+(17)^4 \cos(17x)$ | $f^{(4)}(x_0) =$ | $+(17)^4$ |

see pattern?

**1b.** Find the $N^{\text{th}}$-order Taylor polynomial of $y = f(x)$ about $x_0$ in OPEN form for $N = 0, 1, 2, 3, 4$.

| | | |
|---|---|---|
| $P_0(x) =$ | $1$ | $= 1$ |
| $P_1(x) =$ | $1 + 0x^1$ | $= 1$ |
| $P_2(x) =$ | $1 + 0x^1 + \frac{-(17)^2}{2!}x^2$ | $= 1 - \frac{(17)^2}{2!}x^2$ |
| $P_3(x) =$ | $1 + 0x^1 + \frac{-(17)^2}{2!}x^2 + 0x^3$ | $= 1 - \frac{(17)^2}{2!}x^2$ |
| $P_4(x) =$ | $1 + 0x^1 + \frac{-(17)^2}{2!}x^2 + 0x^3 + \frac{(17)^4}{4!}x^4$ | $= 1 - \frac{(17)^2}{2!}x^2 + \frac{(17)^4}{4!}x^4$ |

**1c.** Find the Taylor series of $y = f(x)$ about $x_0$ in OPEN form.

$$P_\infty(x) = 1 - \frac{(17)^2}{2!}x^2 + \frac{(17)^4}{4!}x^4 - \frac{(17)^6}{6!}x^6 + \frac{(17)^8}{8!}x^8 - \frac{(17)^{10}}{10!}x^{10} + \dots$$

**1d.** Find the Taylor series of $y = f(x)$ about $x_0$ in CLOSED form.

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (17)^{2n}}{(2n)!} x^{2n}$$

**1e.** Find the $n^{\text{th}}$ Taylor coefficient of $y = f(x)$ about $x_0$.

~~[illegible]~~ odd # look like $2n-1 \Rightarrow C_{2n-1} = 0$

even # look like $2n \Rightarrow C_{2n} = \frac{(-1)^n (17)^{2n}}{(2n)!}$

Know $C_n = 0$ if $n$ is odd.

Know $C_n$ has the form $C_n = \frac{(\pm 1)^? (17)^?}{(?)!}$ if $n$ is even.

let's make a chart to figure this out.

all integers, even integers

| $n$ | $2n$ | $C_{2n}$ | |
|---|---|---|---|
| 0 | 0 | $C_0 = 1$ | $= \frac{(-1)^n (17)^{2n}}{(2n)!} = \frac{(-1)^0 (17)^0}{0!} = 1$ |
| 1 | 2 | $C_2 = \frac{-(17)^2}{2!}$ | $= \frac{(-1)^n (17)^{2n}}{(2n)!}$ |
| 2 | 4 | $C_4 = \frac{+(17)^4}{4!}$ | $= \frac{(-1)^n (17)^{2n}}{(2n)!}$ |
| 3 | 6 | $C_6 = \frac{-(17)^6}{6!}$ | $= \frac{(-1)^n (17)^{2n}}{(2n)!}$ |

also works for $n=0$

So $P_\infty(x) = \sum_{n=0}^{\infty} C_{2n} x^{2n}$ ← get only even powers

1f Find the interval of convergence $I$ of the Taylor series of $y = f(x)$ about $x_0$. Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$I = (-\infty, \infty)$ or $\mathbb{R}$

$$a_n = \frac{(-1)^n (17)^{2n}}{(2n)!} x^{2n}$$ factorial $\Rightarrow$ ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (17)^{2(n+1)} x^{2(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{(-1)^n (17)^{2n} x^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{17^{2n+2}}{17^{2n}} \frac{(2n)!}{(2n+2)!} \frac{|x|^{2n+2}}{|x|^{2n}}$$

$$= \lim_{n\to\infty} 17^2 \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)} |x|^2$$

$$= 17^2 |x|^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 17^2 |x|^2 \cdot 0$$

$$= 0 < 1$$

for all $x$'s

1g

Consider the given interval $J$ and fix an $N \in \mathbb{N}$. Find an upper bound for the maximum of $\left|f^{(N+1)}(x)\right|$ on the interval $J$. You answer can have an $N$ in it but it cannot have an: $x$, $x_0$, $c$. (Note that $J$ is a subset of $I$ but Prof. G. might have picked a smaller $J$ than $I$ to make the problem easier.)

$$\max_{c \in J}\left|f^{(N+1)}(c)\right| \le (17)^{N+1}$$

By looking at 1a we see:

$$|f^{(N+1)}(c)| = |\pm (17)^{N+1} \cdot (\cos c \text{ or } \sin c)|$$

$$= (17)^{N+1} \, \underbrace{|\cos c \text{ or } \sin c|}_{1}$$

for all $c \in \mathbb{R}$

$$\le (17)^{N+1}$$

1h

Consider the given interval $J$ and fix an $N \in \mathbb{N}$. For each $x \in J$, find an upper bound for the maximum of $|R_N(x)|$. You answer can have an $N$ and $x$ in it but it cannot have an: $x_0$, $c$.

$$|R_N(x)| \le \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1}$$

$$|R_N(x)| \le \frac{\max_{c \in J} |f^{(N+1)}(c)|}{(N+1)!} |x|^{N+1}$$

1f

$$\le \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1}$$

1i

Carefully show that $f(x) = P_\infty(x)$ for each $x$ in the given interval $J$ by showing that $\lim_{N\to\infty} |R_N(x)| = 0$ for each $x \in J$.

Fix $x \in J$,

$$\underbrace{0 \le |R_N(x)| \underset{\substack{\uparrow \\ \text{by 1g}}}{\le} \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1} \xrightarrow[\text{we want}]{N\to\infty} 0.}_{\text{game plan}}$$

Note $\displaystyle\sum_{N=0}^{\infty} \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1}$ is abs. conv. by ratio test since.

$$a_N = \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1} \quad \text{and so}$$

$$\rho = \lim_{N\to\infty} \left|\frac{a_{N+1}}{a_N}\right| = \lim_{N\to\infty} \left| \frac{(17)^{(N+1)+1} |x|^{(N+1)+1}}{((N+1)+1)!} \cdot \frac{(N+1)!}{17^{N+1} |x|^{N+1}} \right|$$

$$= \lim_{N\to\infty} \frac{17^{N+2}}{17^{N+1}} \frac{(N+1)!}{(N+2)!} \frac{|x|^{N+2}}{|x|^{N+1}} \quad \text{as } \frac{(N+1)!}{(N+2)!} = \frac{(N+1)!}{(N+1)!\,(N+2)}$$

$$= \lim_{N\to\infty} 17 \frac{1}{N+2} |x| \quad \leftarrow x \text{ is fixed, it is } N \text{ that goes to } \infty$$

$$= 17|x| \lim_{N\to\infty} \frac{1}{N+2} = 0 < 1$$

So by $N^{th}$ term test for divergence, $\displaystyle\lim_{N\to\infty} \frac{(17)^{N+1}}{(N+1)!} |x|^{N+1} = 0$.

So the above "game plan" holds true.

So by the Squeeze Theorem,

$f(x) = P_\infty(x)$ for each $x \in J$.