Homework on Taylor/Maclaurin Polynomials and Series

Part 1 — Fill in the box

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

A. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_N(x) =$

B. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_N(x) =$

C. In open form (i.e., with $\ldots$ and without a $\sum$-sign)

$P_\infty(x) =$

D. In closed form (i.e., with a $\sum$-sign and without $\ldots$)

$P_\infty(x) =$

E. We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$R_N(x) = \boxed{}$ for some c between $\boxed{}$ and $\boxed{}$.

F. The formula for c_n is

$c_n =$

Part 2

Do parts (a) - (i) for the following three problems.

(1)	$f(x) = \cos(17x)$	$x_0 = 0$	$J = (-\infty, \infty) = \mathbb{R}$
(2)	$f(x) = (1+x)^{-3}$	$x_0 = 0$	$J = \left(0, \frac{1}{2}\right)$
(3)	$f(x) = e^x$	$x_0 = 17$	$J = (16, 19)$

You might find it easier to do problems (a) - (i) in a different order. Just do what you find easiest.

Use only:

- the definition of Taylor polynominal
- the definition of Taylor series
- the theorem/error-estimate on the N^{th}-Remainder term for Taylor polynomials.

Do **NOT** use a known Taylor Series (i.e., do not use methods from section 10.10).

a. Find the following. Note the first column are functions of x and the second column are numbers.

$f^{(0)}(x) =$	$f^{(0)}(x_0) =$
$f^{(1)}(x) =$	$f^{(1)}(x_0) =$
$f^{(2)}(x) =$	$f^{(2)}(x_0) =$
$f^{(3)}(x) =$	$f^{(3)}(x_0) =$
$f^{(4)}(x) =$	$f^{(4)}(x_0) =$

b. Find the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0 in OPEN form for $N = 0, 1, 2, 3, 4$.

$P_0(x) =$
$P_1(x) =$
$P_2(x) =$
$P_3(x) =$
$P_4(x) =$

c. Find the Taylor series of $y = f(x)$ about x_0 in OPEN form.

$$P_\infty(x) \ =$$

d. Find the Taylor series of $y = f(x)$ about x_0 in CLOSED form.

$$P_\infty(x) \ =$$

e. Find the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

$$c_n \ =$$

f. Find the interval of convergence I of the Taylor series of $y = f(x)$ about x_0. Recall, the interval of convergence is the set of points for which the series converges, either absolutely or conditionally. (Hint: use the ratio or root test and then check the endpoints.)

$I =$

g. Consider the given interval J and fix an $N \in \mathbb{N}$. Find an upper bound for the maximum of $\left|f^{(N+1)}(x)\right|$ on the interval J. You answer can have an N in it but it cannot have an: x, x_0, c. (Note that J is a subset of I but Prof. G. might have picked a smaller J than I to make the problem easier.)

$$\max_{c \in J} \left|f^{(N+1)}(c)\right| \ \leq$$

h. Consider the given interval J and fix an $N \in \mathbb{N}$. For each $x \in J$, find an upper bound for the maximum of $|R_N(x)|$. You answer can have an N and x in it but it cannot have an: x_0, c.

$$|R_N(x)| \ \leq$$

i. Carefully show that $f(x) = P_\infty(x)$ for each x in the given interval J by showing that $\lim_{N\to\infty} |R_N(x)| = 0$ for each $x \in J$.