$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right)=\sum_{n=1}^{\infty}\frac{1}{n^p}$$
which converges if $p > 1$. Thus, by the Limit Comparison Test, if $\sum_{n=1}^{\infty} b_n$ converges for $p > 1$, so does $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$ also converges. For $p \le 1$, since $1+\frac{1}{2^p}+\ldots+\frac{1}{n^p} > 1$,

$\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right) > \frac{1}{n^p}$. Hence, since $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p \le 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right)$ also diverges. The series converges for $p > 1$ and diverges for $p \le 1$.

10.5 Concepts Review

1. $\lim_{n\to\infty} a_n = 0$

3. the alternating harmonic series

Problem Set 10.5

1. $a_n = \frac{2}{3n+1}$; $\frac{2}{3n+1} > \frac{2}{3n+4}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty}\frac{2}{3n+1} = 0$. $S_9 \approx 0.363$. The error made by using S_9 is not more than $a_{10} \approx 0.065$.

3. $a_n = \frac{1}{\ln(n+1)}$; $\frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty}\frac{1}{\ln(n+1)} = 0$. $S_9 \approx 1.137$. The error made by using S_9 is not more than $a_{10} \approx 0.417$.

5. $a_n = \frac{\ln n}{n}$; $\frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$ is equivalent to $\ln\frac{n^{n+1}}{(n+1)^n} > 0$ or $\frac{n^{n+1}}{(n+1)^n} > 1$ which is true for $n > 2$. $S_9 \approx -0.041$. The error made by using S_9 is not more than $a_{10} \approx 0.230$.

7. $\frac{|u_{n+1}|}{|u_n|} = \frac{\left|\left(-\frac{3}{4}\right)^{n+1}\right|}{\left|\left(-\frac{3}{4}\right)^n\right|} = \frac{3}{4} < 1$, so the series converges absolutely.

9. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}$; $\lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} < 1$, so the series converges absolutely.

11. $n(n+1) = n^2 + n > n^2$ for all $n > 0$, thus $\frac{1}{n(n+1)} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges since $2 > 1$, thus $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n(n+1)}$ converges absolutely.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which converges since $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. The series is conditionally convergent since $\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

15. $\lim_{n\to\infty}\frac{n}{10n+1} = \frac{1}{10} \ne 0$. Thus the sequence of partial sums does not converge; the series diverges.

17. $\lim_{n\to\infty}\frac{1}{n\ln n} = 0$; $\frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$ is equivalent to $(n+1)^{n+1} > n^n$ which is true for all $n > 0$ so $a_n > a_{n+1}$. The alternating series converges.
$\sum_{n=2}^{\infty}|u_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$; $\frac{1}{x\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$.
Using $u = \ln x$, $du = \frac{1}{x}dx$,

$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = \left[\ln|u| \right]_{\ln 2}^{\infty} = \infty$. Thus, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ is conditionally convergent.

19. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}} = \frac{n^4}{2(n+1)^4}$;

$\lim_{n\to\infty} \frac{n^4}{2(n+1)^4} = \frac{1}{2} < 1$.

The series is absolutely convergent.

21. $a_n = \frac{n}{n^2+1}$; $\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$ is equivalent to $n^2 + n - 1 > 0$, which is true for $n > 1$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, hence the alternating series converges. Let $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1; 0 < 1 < \infty$. Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges. The series is conditionally convergent.

23. $\cos n\pi = (-1)^n = \frac{1}{(-1)}(-1)^{n+1}$ so the series is $-1\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, -1 times the alternating harmonic series. The series is conditionally convergent.

25. $|\sin n| \le 1$ for all n, so

$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges since $\frac{3}{2} > 1$. Thus the series is absolutely convergent.

27. $a_n = \frac{1}{\sqrt{n(n+1)}}$; $\frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+2)}}$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} = 0$ so the alternating series converges.

Let $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$;

$0 < 1 < \infty$.

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ also diverges.

The series is conditionally convergent.

29. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n^2}$; $\lim_{n\to\infty} \frac{3^{n+1}}{n^2} \neq 0$, so the series diverges.

31. Suppose $\sum |a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum (|a_n| + a_n)$ converges since $0 \le |a_n| + a_n \le 2|a_n|$. By the linearity of convergent series, $\sum a_n = \sum (|a_n| + a_n) - \sum |a_n|$ converges, which is a contradiction.

33. The positive-term series is

$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$.

$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges since the harmonic series diverges.

Thus, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

The negative-term series is

$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \ldots = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges, since the harmonic series diverges.

35. a. $1 + \frac{1}{3} \approx 1.33$

b. $1 + \frac{1}{3} - \frac{1}{2} \approx 0.833$

c. $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \approx 1.38$

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} \approx 1.13$

37. Written response

39. Consider $1-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{9}+\ldots$

It is clear that $\lim_{n\to\infty} a_n = 0$. Pairing successive terms, we obtain $\frac{1}{n}-\frac{1}{n^2}=\frac{n-1}{n^2}>0$ for $n>1$.

Let $a_n=\frac{n-1}{n^2}$ and $b_n=\frac{1}{n}$. Then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2-n}{n^2}=1;\ 0<1<\infty$.

Thus, since $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$ also diverges.

41. Note that $(a_k+b_k)^2\ge 0$ and $(a_k-b_k)^2\ge 0$ for all k. Thus, $a_k^2\pm 2a_kb_k+b_k^2\ge 0$, or

$a_k^2+b_k^2\ge \pm 2a_kb_k$ for all k, and

$a_k^2+b_k^2\ge 2|a_kb_k|$. Since $\sum_{k=1}^{\infty}a_k^2$ and $\sum_{k=1}^{\infty}b_k^2$ both converge, $\sum_{k=1}^{\infty}(a_k^2+b_k^2)$ also converges, and by the Comparison Test, $\sum_{k=1}^{\infty}2|a_kb_k|$ converges.

Hence, $\sum_{k=1}^{\infty}|a_kb_k|=\frac{1}{2}\sum_{k=1}^{\infty}2|a_kb_k|$ converges, i.e., $\sum_{k=1}^{\infty}a_kb_k$ converges absolutely.

43. Consider the graph of $\frac{|\sin x|}{x}$ on the interval $[k\pi, (k+1)\pi]$.

Note that for $k\pi+\frac{\pi}{6}\le x\le k\pi+\frac{5\pi}{6}$, $\frac{1}{2}\le|\sin x|$ while $\frac{1}{\left(k+\frac{5}{6}\right)\pi}\le\frac{1}{x}$. Thus on $\left[\left(k+\frac{1}{6}\right)\pi,\left(k+\frac{5}{6}\right)\pi\right]$

$\frac{1}{2\left(k+\frac{5}{6}\right)\pi}=\frac{1}{\left(2k+\frac{5}{3}\right)\pi}\le\frac{|\sin x|}{x}$, so $\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx\ge\int_{(k+1/6)\pi}^{(k+5/6)\pi}\frac{|\sin x|}{x}dx\ge\frac{1}{\left(2k+\frac{5}{3}\right)\pi}\int_{(k+1/6)\pi}^{(k+5/6)\pi}dx=\frac{1}{3k+\frac{5}{2}}$.

Hence, $\int_{\pi}^{\infty}\frac{|\sin x|}{x}dx\ge\sum_{k=1}^{\infty}\frac{1}{3k+\frac{5}{2}}$. Let $a_k=\frac{1}{3k+\frac{5}{2}}$ and $b_k=\frac{1}{k}$.

$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{k}{3k+\frac{5}{2}}=\lim_{k\to\infty}\frac{1}{3+\frac{5}{2k}}=\frac{1}{3};\ 0<\frac{1}{3}<\infty$. Thus, since $\sum_{k=1}^{\infty}b_k=\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, $\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}\frac{1}{3k+\frac{5}{2}}$ also diverges. Hence, $\int_{\pi}^{\infty}\frac{|\sin x|}{x}dx$ also diverges and adding $\int_{0}^{\pi}\frac{|\sin x|}{x}dx$ will not affect its divergence.

45. $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]\left(\frac{1}{n}\right)$

This is a Riemann sum for the function $f(x)=\frac{1}{x}$ from $x=1$ to 2 where $\Delta x=\frac{1}{n}$.

$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{1+\frac{k}{n}}\left(\frac{1}{n}\right)\right]=\int_1^2\frac{1}{x}dx=\ln 2$