Concepts Review

1. If $a_n \geq 0$ for all n, the alternating series $a_1 - a_2 + a_3 - \cdots$ will converge provided that the terms are decreasing in size and _____.

2. If $\sum |u_k|$ converges, we say that the series $\sum u_k$ converges _____; if $\sum u_k$ converges, but $\sum |u_k|$ diverges, we say that $\sum u_k$ converges _____.

3. The premier example of a conditionally convergent series is _____.

4. The terms of an absolutely convergent series may be _____ at will without affecting its convergence or its sum.

Problem Set 10.5

In Problems 1–6, show that each alternating series converges and then estimate the error made by using the partial sum S_6 as an approximation to the sum S of the series (see Examples 1–3).

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{3n+1}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$

In Problems 7–12, show that each series converges absolutely.

7. $\displaystyle\sum_{n=1}^{\infty} \left(-\tfrac{3}{4}\right)^n$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n}}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^n}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$

In Problems 13–30, classify each series as absolutely convergent, conditionally convergent, or divergent.

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n}$

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n^{1.1}}$

15. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{10n+1}$

16. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{10n^{1.1}+1}$

17. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{n \ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(1+\sqrt{n})}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^4}{2^n}$

20. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{\sqrt{n^2-1}}$

21. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

22. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n-1}{n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2}$

25. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{\sin n}{n\sqrt{n}}$

26. $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

27. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n(n+1)}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n^2}$

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \sin\frac{\pi}{n}$

31. Prove that if $\sum a_n$ diverges, so does $\sum |a_n|$.

32. Give an example of two series $\sum a_n$ and $\sum b_n$, both convergent, such that $\sum a_n b_n$ diverges.

33. Show that the positive terms of the alternating harmonic series form a divergent series. Show the same for the negative terms.

34. Show that the results in Problem 33 hold for any conditionally convergent series.

35. Show that the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

(whose sum is actually $\ln 2 \approx 0.69$) can be rearranged to converge to 1.3 by using the following steps.

(a) Take enough of the positive terms $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ to just exceed 1.3.

(b) Now add enough of the negative terms $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$ so that the partial sum S_n falls just below 1.3.

(c) Add just enough more positive terms to again exceed 1.3, and so on.

C **36.** Use your calculator to help you to find the first 20 terms of the series described in Problem 35. Calculate S_{20}.

37. Explain why a conditionally convergent series can be rearranged to converge to any given number.

38. Show that a conditionally convergent series can be rearranged so as to diverge.

39. Show that $\lim_{n\to\infty} a_n = 0$ is not sufficient to guarantee the convergence of the alternating series $\sum (-1)^{n+1} a_n$. *Hint*: Alternate the terms of $\sum 1/n$ and $\sum -1/n^2$.