The upper rectangles, which extend to $n + 1$ on the right, have area $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$. These rectangles are above the curve $y=\frac{1}{x}$ from $x = 1$ to $x = n + 1$. Thus,

$$\int_1^{n+1}\frac{1}{x}dx=[\ln x]_1^{n+1}=\ln(n+1)-\ln 1=\ln(n+1)$$
$$<1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

The lower (shaded) rectangles have area $\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$. These rectangles lie below the curve $y=\frac{1}{x}$ from $x = 1$ to $x = n$. Thus

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<\int_1^n\frac{1}{x}dx=\ln n, \text{ so}$$
$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<1+\ln n.$$

31. $\{B_n\}$ is a nondecreasing sequence that is bounded above, thus by the Monotone Sequence Theorem (Theorem D of Section 10.1), $\lim_{n\to\infty} B_n$ exists.
The rationality of γ is a famous unsolved problem.

33. $\gamma+\ln(n+1)>20 \Rightarrow \ln(n+1)>20-\gamma\approx 19.4228$
$\Rightarrow n+1>e^{19.4228}\approx 272,404,867$
$\Rightarrow n>272,404,866$

35. Every time n is incremented by 1, a positive amount of area is added, thus $\{A_n\}$ is an increasing sequence.
Each curved region has horizontal width 1, and can be moved into the heavily outlined triangle without any overlap. This can be done by shifting the nth shaded region, which goes from $(n, f(n))$ to $(n + 1, f(n + 1))$, as follows:
shift $(n + 1, f(n + 1))$ to $(2, f(2))$ and $(n, f(n))$ to $(1, f(2)–[f(n + 1) – f(n)])$.
The slope of the line forming the bottom of the shaded region between $x = n$ and $x = n + 1$ is

$$\frac{f(n+1)-f(n)}{(n+1)-n}=f(n+1)-f(n)>0$$

since f is increasing. By the Mean Value Theorem, $f(n+1)-f(n)=f'(c)$ for some c in $(n, n + 1)$. Since f is concave down, $n < c < n + 1$ means that $f'(c)<f'(b)$ for all b in $[1, n]$. Thus, the nth shaded region will not overlap any other shaded region when shifted into the heavily outlined triangle. Thus, the area of all of the shaded regions is less than or equal to the area of the heavily outlined triangle, so $\lim_{n\to\infty} A_n$ exists.

10.4 Concepts Review

1. $0\le a_k\le b_k$

3. $\rho<1;\ \rho>1;\ \rho=1$

Problem Set 10.4

1. $a_n=\frac{n}{n^2+2n+3};\ b_n=\frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2+2n+3}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{3}{n^2}}=1;$$

$0<1<\infty$

$\sum_{n=1}^{\infty}b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty}a_n$ diverges.

3. $a_n=\frac{1}{n\sqrt{n+1}}=\frac{1}{\sqrt{n^3+n^2}};\ b_n=\frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+n^2}}=\lim_{n\to\infty}\sqrt{\frac{n^3}{n^3+n^2}}$$
$$=\lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}=1;\ 0<1<\infty$$

$\sum_{n=1}^{\infty}b_n$ converges $\Rightarrow \sum_{n=1}^{\infty}a_n$ converges

5. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{8^{n+1}n!}{(n+1)!8^n}=\lim_{n\to\infty}\frac{8}{n+1}=0<1$
The series converges.

7. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!n^{100}}{(n+1)^{100}n!}=\lim_{n\to\infty}\frac{n^{100}}{(n+1)^{99}}$

$$=\lim_{n\to\infty}\frac{n}{\left(\frac{n+1}{n}\right)^{99}}=\infty \text{ since } \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{99}=1$$

The series diverges.

9. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^3(2n)!}{(2n+2)!n^3}$

$$=\lim_{n\to\infty}\frac{(n+1)^3}{(2n+2)(2n+1)n^3}=\lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{4n^5+6n^4+2n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{3}{n^3}+\frac{3}{n^4}+\frac{1}{n^5}}{4+\frac{6}{n}+\frac{2}{n^2}} = 0 < 1$$

The series converges.

11. $\lim_{n\to\infty} \frac{n}{n+200} = \lim_{n\to\infty} \frac{1}{1+\frac{200}{n}} = 1 \neq 0$

The series diverges; nth-Term Test

13. $a_n = \frac{n+3}{n^2\sqrt{n}}; b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{5/2}+3n^{3/2}}{n^{5/2}} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1} = 1;$$

$0 < 1 < \infty$. $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

15. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 n!}{(n+1)!n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{(n+1)n^2}$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^3+n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}}{1+\frac{1}{n}} = 0 < 1$$

The series converges; Ratio Test

17. $a_n = \frac{4n^3+3n}{n^5-4n^2+1}; b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^5+3n^3}{n^5-4n^2+1} = \lim_{n\to\infty} \frac{4+\frac{3}{n^2}}{1-\frac{4}{n^3}+\frac{1}{n^5}} = 4;$$

$0 < 4 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

19. $a_n = \frac{1}{n(n+1)} = \frac{1}{n^2+n}; b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1;$$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

21. $a_n = \frac{n+1}{n(n+2)(n+3)} = \frac{n+1}{n^3+5n^2+6n}; b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+5n^2+6n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = 1;$$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

23. $a_n = \frac{n}{3^n}; \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)3^n}{3^{n+1}n}$

$$= \lim_{n\to\infty} \frac{n+1}{3n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges; Ratio Test

25. $a_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}; \frac{1}{x^{3/2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx = \left[-\frac{2}{\sqrt{x}}\right]_1^{\infty} = 0+2 = 2 < \infty$$

The series converges; Integral Test

27. $0 \le \sin^2 n \le 1$ for all n, so

$2 \le 2+\sin^2 n \le 3 \Rightarrow \frac{1}{2} \ge \frac{1}{2+\sin^2 n} \ge \frac{1}{3}$ for all n.

Thus, $\lim_{n\to\infty} \frac{1}{2+\sin^2 n} \neq 0$ and the series diverges; nth-Term Test

29. $-1 \le \cos n \le 1$ for all n, so

$3 \le 4+\cos n \le 5 \Rightarrow \frac{3}{n^3} \le \frac{4+\cos n}{n^3} \le \frac{5}{n^3}$ for all n.

$\sum_{n=1}^{\infty} \frac{5}{n^3}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{4+\cos n}{n^3}$ converges; Comparison Test

31. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}(2n)!}{(2n+2)!n^n}$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2(n+1)(2n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{2(2n+1)n^n} = \lim_{n\to\infty} \left[\frac{1}{4n+2}\left(\frac{n+1}{n}\right)^n\right]$$

$$= \left[\lim_{n\to\infty} \frac{1}{4n+2}\right]\left[\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\right] = 0 \cdot e = 0 < 1$$

(The limits can be separated since both limits exist.) The series converges; Ratio Test

33. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(4^{n+1}+n+1)n!}{(n+1)!(4^n+n)}$

$$= \lim_{n\to\infty} \frac{4^{n+1}+n+1}{(n+1)(4^n+n)} = \lim_{n\to\infty} \frac{4+\frac{n}{4^n}+\frac{1}{4^n}}{(n+1)\left(1+\frac{n}{4^n}\right)}$$

$$= \lim_{n\to\infty} \frac{4 + \frac{n}{4^n} + \frac{1}{4^n}}{1 + n + \frac{n}{4^n} + \frac{n^2}{4^n}} = 0$$

since $\lim_{n\to\infty} \frac{n^2}{4^n} = 0$, $\lim_{n\to\infty} \frac{n}{4^n} = 0$, and

$\lim_{n\to\infty} \frac{1}{4^n} = 0$. The series converges; Ratio Test

35. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Thus, there is some positive integer N such that $0 < a_n < 1$ for all $n \geq N$. $a_n < 1 \Rightarrow a_n{}^2 < a_n$, thus $\sum_{n=N}^{\infty} a_n{}^2 < \sum_{n=N}^{\infty} a_n$. Hence $\sum_{n=N}^{\infty} a_n{}^2$ converges, and $\sum a_n{}^2$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

37. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then there is some positive integer N such that $0 \leq \frac{a_n}{b_n} < 1$ for all $n \geq N$. Thus, for $n \geq N$, $a_n < b_n$. By the Comparison Test, since $\sum_{n=N}^{\infty} b_n$ converges, $\sum_{n=N}^{\infty} a_n$ also converges. Thus, $\sum a_n$ converges since adding a finite number of terms will not affect the convergence or divergence of a series.

39. If $\lim_{n\to\infty} na_n = 1$ then there is some positive integer N such that $a_n \geq 0$ for all $n \geq N$, Let $b_n = \frac{1}{n}$, so $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} na_n = 1 < \infty$.

Since $\sum_{n=N}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=N}^{\infty} a_n$ diverges by the Limit Comparison Test.

Thus $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

41. Suppose that $\lim_{n\to\infty} (a_n)^{1/n} = R$ where $a_n > 0$.

If $R < 1$, there is some number r with $R < r < 1$ and some positive integer N such that $\left|(a_n)^{1/n} - R\right| < r - R$ for all $n \geq N$. Thus,

$R - r < (a_n)^{1/n} - R < r - R$ or

$-r < (a_n)^{1/n} < r < 1$. Since $a_n > 0$,

$0 < (a_n)^{1/n} < r$ and $0 < a_n < r^n$ for all $n \geq N$.

Thus, $\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} r^n$, which converges since $|r| < 1$. Thus, $\sum_{n=N}^{\infty} a_n$ converges so $\sum a_n$ also converges.

If $R > 1$, there is some number r with $1 < r < R$ and some positive integer N such that $\left|(a_n)^{1/n} - R\right| < R - r$ for all $n \geq N$. Thus,

$r - R < (a_n)^{1/n} - R < R - r$ or

$r < (a_n)^{1/n} < 2R - r$ for all $n \geq N$. Hence

$r^n < a_n$ for all $n \geq N$, so $\sum_{n=N}^{\infty} r^n < \sum_{n=N}^{\infty} a_n$, and since $\sum_{n=N}^{\infty} r^n$ diverges ($r > 1$), $\sum_{n=N}^{\infty} a_n$ also diverges, so $\sum a_n$ diverges.

43. a. $\ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots$
$+ (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$
$= -\ln 1 + \ln(n+1) = \ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$

Since the partial sums are unbounded, the series diverges.

b. $\ln \frac{(n+1)^2}{n(n+2)} = 2\ln(n+1) - \ln n - \ln(n+2)$

$S_n = (2\ln 2 - \ln 1 - \ln 3) + (2\ln 3 - \ln 2 - \ln 4)$
$+ (2\ln 4 - \ln 3 - \ln 5) + \ldots$
$+ (2\ln n - \ln(n-1) - \ln(n+1))$
$+ (2\ln(n+1) - \ln n - \ln(n+2))$
$= \ln 2 - \ln 1 + \ln(n+1) - \ln(n+2)$
$= \ln 2 + \ln \frac{n+1}{n+2}$

$\lim_{n\to\infty} S_n = \ln 2 + \lim_{n\to\infty} \ln \frac{n+1}{n+2} = \ln 2$

Since the partial sums converge, the series converges.

c. $\left(\frac{1}{\ln x}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$, thus $\sum_{n=2}^{\infty} \left(\frac{1}{\ln n}\right)^{\ln n}$

converges if and only if $\int_2^\infty \left(\frac{1}{\ln x}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\int_2^\infty \left(\frac{1}{\ln x}\right)^{\ln x} dx = \int_{\ln 2}^\infty \left(\frac{1}{u}\right)^u e^u du = \int_{\ln 2}^\infty \left(\frac{e}{u}\right)^u du$$

This integral converges if and only if the associated series, $\sum_{n=1}^\infty \left(\frac{e}{n}\right)^n$ converges. With $a_n = \left(\frac{e}{n}\right)^n$, the Root Test (Problem 41) gives

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{e}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{e}{n} = 0 < 1$$

Thus, $\sum_{n=1}^\infty \left(\frac{e}{n}\right)^n$ converges, so $\int_{\ln 2}^\infty \left(\frac{e}{u}\right)^u du$ converges, whereby $\sum_{n=2}^\infty \frac{1}{(\ln n)^{\ln n}}$ converges.

d. $\left(\frac{1}{\ln(\ln x)}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[3, \infty)$, thus $\sum_{n=3}^\infty \left(\frac{1}{\ln(\ln n)}\right)^{\ln n}$ converges if and only if $\int_3^\infty \left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\int_3^\infty \left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$$

$$= \int_{\ln 3}^\infty \left(\frac{1}{\ln u}\right)^u e^u du = \int_{\ln 3}^\infty \left(\frac{e}{\ln u}\right)^u du.$$

This integral converges if and only if the associated series, $\sum_{n=2}^\infty \left(\frac{e}{\ln n}\right)^n$ converges.

With $a_n = \left(\frac{e}{\ln n}\right)^n$, the Root Test (Problem 41) gives

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{e}{\ln n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{e}{\ln n} = 0 < 1$$

Thus, $\sum_{n=2}^\infty \left(\frac{e}{\ln n}\right)^n$ converges, so $\int_{\ln 3}^\infty \left(\frac{e}{\ln u}\right)^u du$ converges, whereby $\sum_{n=3}^\infty \frac{1}{(\ln(\ln n))^{\ln n}}$ converges.

e. $a_n = 1/n$; $b_n = 1/(\ln n)^4$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n}{1/(\ln n)^4} = \lim_{n\to\infty} \frac{(\ln n)^4}{n}$$

$$= \lim_{n\to\infty} \frac{4(\ln n)^3(1/n)}{1} = \lim_{n\to\infty} \frac{4(\ln n)^3}{n}$$

$$= \lim_{n\to\infty} \frac{12(\ln n)^2(1/n)}{1} = \lim_{n\to\infty} \frac{12(\ln n)^2}{n}$$

$$= \lim_{n\to\infty} \frac{24(\ln n)}{n} = \lim_{n\to\infty} \frac{24(1/n)}{1}$$

$$= \lim_{n\to\infty} \frac{24}{n} = 0$$

$\sum_{n=2}^\infty \frac{1}{n}$ diverges $\Rightarrow \sum_{n=2}^\infty \frac{1}{(\ln n)^4}$ diverges

f. $\left(\frac{\ln x}{x}\right)^2$ is continuous, positive, and nonincreasing on $[3, \infty)$. Using integration by parts twice,

$$\int_3^\infty \left(\frac{\ln x}{x}\right)^2 dx = \left[-\frac{(\ln x)^2}{x}\right]_3^\infty + \int_3^\infty \frac{2\ln x}{x^2} dx$$

$$= \left[-\frac{(\ln x)^2}{x}\right]_3^\infty + \left[-\frac{2\ln x}{x}\right]_3^\infty + \int_3^\infty \frac{2}{x^2} dx$$

$$= \left[-\frac{(\ln x)^2}{x} - \frac{2\ln x}{x} - \frac{2}{x}\right]_3^\infty \approx 1.8 < \infty$$

Thus, $\sum_{n=3}^\infty \left(\frac{\ln x}{x}\right)^2$ converges.

45. Let $a_n = \frac{1}{n^p}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p}\right)$ and $b_n = \frac{1}{n^p}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right)=\sum_{n=1}^{\infty}\frac{1}{n^p}$$

which converges if $p > 1$. Thus, by the Limit Comparison Test, if $\sum_{n=1}^{\infty} b_n$ converges for $p > 1$, so does $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$ also converges. For $p \le 1$, since $1+\frac{1}{2^p}+\ldots+\frac{1}{n^p} > 1$,

$\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right) > \frac{1}{n^p}$. Hence, since $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p \le 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right)$ also diverges. The series converges for $p > 1$ and diverges for $p \le 1$.

10.5 Concepts Review

1. $\lim_{n\to\infty} a_n = 0$

3. the alternating harmonic series

Problem Set 10.5

1. $a_n = \frac{2}{3n+1}; \frac{2}{3n+1} > \frac{2}{3n+4}$, so $a_n > a_{n+1}$;

   $\lim_{n\to\infty}\frac{2}{3n+1} = 0$. $S_9 \approx 0.363$. The error made by using S_9 is not more than $a_{10} \approx 0.065$.

3. $a_n = \frac{1}{\ln(n+1)}; \frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$, so

   $a_n > a_{n+1}$;

   $\lim_{n\to\infty}\frac{1}{\ln(n+1)} = 0$. $S_9 \approx 1.137$. The error made by using S_9 is not more than $a_{10} \approx 0.417$.

5. $a_n = \frac{\ln n}{n}; \frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$ is equivalent to

   $\ln\frac{n^{n+1}}{(n+1)^n} > 0$ or $\frac{n^{n+1}}{(n+1)^n} > 1$ which is true for $n > 2$. $S_9 \approx -0.041$. The error made by using S_9 is not more than $a_{10} \approx 0.230$.

7. $\frac{|u_{n+1}|}{|u_n|} = \frac{\left|\left(-\frac{3}{4}\right)^{n+1}\right|}{\left|\left(-\frac{3}{4}\right)^{n}\right|} = \frac{3}{4} < 1$, so the series converges absolutely.

9. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}; \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} < 1$, so the series converges absolutely.

11. $n(n+1) = n^2 + n > n^2$ for all $n > 0$, thus

    $\frac{1}{n(n+1)} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges since $2 > 1$, thus

    $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n(n+1)}$ converges absolutely.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which converges

    since $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. The series is

    conditionally convergent since $\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

15. $\lim_{n\to\infty}\frac{n}{10n+1} = \frac{1}{10} \ne 0$. Thus the sequence of partial sums does not converge; the series diverges.

17. $\lim_{n\to\infty}\frac{1}{n\ln n} = 0; \frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$ is equivalent to $(n+1)^{n+1} > n^n$ which is true for all $n > 0$ so $a_n > a_{n+1}$. The alternating series converges.

    $\sum_{n=2}^{\infty}|u_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln n}; \frac{1}{x\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

    Using $u = \ln x$, $du = \frac{1}{x}dx$,