① comparison test
② limit comparison test
③ ratio test
④ root test

4. If the tests above do not work, try the Ordinary Comparison Test, the Integral Test, or the Bounded Sum Test.
5. Some series require a clever manipulation or a neat trick to determine convergence or divergence.

Concepts Review

1. The Ordinary Comparison Test says that if ________ and if Σb_k converges, then Σa_k also converges.

2. Assume that $a_k \geq 0$ and $b_k > 0$. The Limit Comparison Test says that if $0 <$ ________ $< \infty$ then Σa_k and Σb_k converge or diverge together.

3. Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. The Ratio Test says that a series Σa_k of positive terms converges if ________, diverges if ________, and may do either if ________.

4. $\Sigma(3^k/k!)$ is an obvious candidate for the ________ Test, whereas $\Sigma k/(k^3 - k - 1)$ is an obvious candidate for the ________ Test.

Problem Set 10.4

In Problems 1–4, use the Limit Comparison Test to determine convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 2n + 3}$

2. $\sum_{n=1}^{\infty} \frac{3n + 1}{n^3 - 4}$

3. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n + 1}}$

4. $\sum_{n=1}^{\infty} \frac{\sqrt{2n + 1}}{n^2}$

In Problems 5–10, use the Ratio Test to determine convergence or divergence.

5. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$

6. $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$

7. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$

8. $\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n$

9. $\sum_{n=1}^{\infty} \frac{n^3}{(2n)!}$

10. $\sum_{k=1}^{\infty} \frac{3^k + k}{k!}$

In Problems 11–34, determine convergence or divergence for each of the series. Indicate the test you use.

11. $\sum_{n=1}^{\infty} \frac{n}{n + 200}$

12. $\sum_{n=1}^{\infty} \frac{n!}{5 + n}$

13. $\sum_{n=1}^{\infty} \frac{n + 3}{n^2\sqrt{n}}$

14. $\sum_{n=1}^{\infty} \frac{\sqrt{n + 1}}{n^2 + 1}$

15. $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

16. $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$

17. $\sum_{n=1}^{\infty} \frac{4n^3 + 3n}{n^5 - 4n^2 + 1}$

18. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{3^n}$

19. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$

Hint: $a_n = \frac{1}{n(n + 1)}$.

20. $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \frac{4}{5^2} + \cdots$

21. $\frac{2}{1 \cdot 3 \cdot 4} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{4}{3 \cdot 5 \cdot 6} + \frac{5}{4 \cdot 6 \cdot 7} + \cdots$

22. $\frac{1}{1^2 + 1} + \frac{2}{2^2 + 1} + \frac{3}{3^2 + 1} + \frac{4}{4^2 + 1} + \cdots$

23. $\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$

24. $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$

25. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$

26. $\frac{\ln 2}{2^2} + \frac{\ln 3}{3^2} + \frac{\ln 4}{4^2} + \frac{\ln 5}{5^2} + \cdots$

27. $\sum_{n=1}^{\infty} \frac{1}{2 + \sin^2 n}$

28. $\sum_{n=1}^{\infty} \frac{5}{3^n + 1}$

29. $\sum_{n=1}^{\infty} \frac{4 + \cos n}{n^3}$

30. $\sum_{n=1}^{\infty} \frac{5^{2n}}{n!}$

31. $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$

32. $\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^n$

33. $\sum_{n=1}^{\infty} \frac{4^n + n}{n!}$

34. $\sum_{n=1}^{\infty} \frac{n}{2 + n5^n}$

35. Let $a_n > 0$ and suppose that Σa_n converges. Prove that Σa_n^2 converges.

36. Prove that $\lim_{n\to\infty} (n!/n^n) = 0$ by considering the series $\Sigma n!/n^n$. *Hint*: Example 7, followed by nth-Term Test.

37. Prove that if $a_n \geq 0, b_n > 0, \lim_{n\to\infty} a_n/b_n = 0$, and Σb_n converges then Σa_n converges.

38. Prove that if $a_n \geq 0, b_n > 0, \lim_{n\to\infty} a_n/b_n = \infty$, and Σb_n diverges then Σa_n diverges.

39. Suppose that $\lim_{n\to\infty} na_n = 1$. Prove that Σa_n diverges.

40. Prove that if Σa_n is a convergent series of positive terms then $\Sigma \ln(1 + a_n)$ converges.

41. **Root Test** Prove that if $a_n > 0$ and $\lim_{n\to\infty} (a_n)^{1/n} = R$ then Σa_n converges if $R < 1$ and diverges if $R > 1$.

42. Test for convergence or divergence using the Root Test.

(a) $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^n$ (b) $\sum_{n=1}^{\infty}\left(\frac{n}{3n+2}\right)^n$

(c) $\sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{n}\right)^n$

43. Test for convergence or divergence. In some cases, a clever manipulation using the properties of logarithms will simplify the problem.

(a) $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$ (b) $\sum_{n=1}^{\infty}\ln\left[\frac{(n+1)^2}{n(n+2)}\right]$

(c) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ (d) $\sum_{n=3}^{\infty}\frac{1}{[\ln(\ln n)]^{\ln n}}$

(e) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^4}$ (f) $\sum_{n=1}^{\infty}\left[\frac{\ln n}{n}\right]^2$

EXPL **44.** Let $p(n)$ and $q(n)$ be polynomials in n with nonnegative coefficients. Give simple conditions that determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$.

EXPL **45.** Give conditions on p that determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$.

46. Test for convergence or divergence.

(a) $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ (b) $\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

(c) $\sum_{n=1}^{\infty}\sqrt{n}\left[1-\cos\left(\frac{1}{n}\right)\right]$

Answers to Concepts Review: **1.** $0 \le a_k \le b_k$ **2.** $\lim_{k\to\infty}(a_k/b_k)$ **3.** $\rho < 1$; $\rho > 1$; $\rho = 1$ **4.** Ratio; Limit Comparison

10.5 Alternating Series, Absolute Convergence, and Conditional Convergence

In the last two sections, we considered series of nonnegative terms. Now we remove that restriction, allowing some terms to be negative. In particular, we study **alternating series**, that is, series of the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

where $a_n > 0$ for all n. An important example is the **alternating harmonic series**

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

We have seen that the harmonic series diverges; we shall soon see that the *alternating* harmonic series converges.

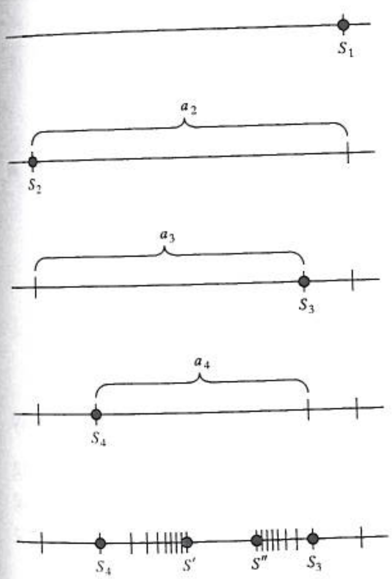

Figure 1

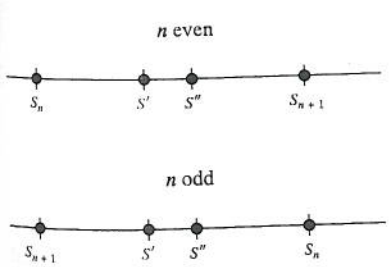

Figure 2

A Convergence Test Let us suppose that the sequence $\{a_n\}$ is decreasing; that is, $a_{n+1} < a_n$ for all n. Also, let S_n have its usual meaning. Thus, for the alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, we have

$$S_1 = a_1$$
$$S_2 = a_1 - a_2 = S_1 - a_2$$
$$S_3 = a_1 - a_2 + a_3 = S_2 + a_3$$
$$S_4 = a_1 - a_2 + a_3 - a_4 = S_3 - a_4$$

and so on. A geometric interpretation of these partial sums is shown in Figure 1. Note that the even-numbered terms $S_2, S_4, S_6, \ldots$ are increasing and bounded above and hence must converge to a limit, call it S'. Similarly, the odd-numbered terms $S_1, S_3, S_5, \ldots$ are decreasing and bounded below. They also converge, say to S''.

Both S' and S'' are between S_n and S_{n+1} for all n (see Figure 2), and so

$$|S'' - S'| \le |S_{n+1} - S_n| = a_{n+1}$$

Thus, the condition $a_{n+1} \to 0$ as $n \to \infty$ will guarantee that $S' = S''$ and, consequently, the convergence of the series to their common value, which we call S. Finally, we note that, since S is between S_n and S_{n+1},

$$|S - S_n| \le |S_{n+1} - S_n| = a_{n+1}$$

That is, the error made by using S_n as an approximation to the sum S of the whole series is not more than the magnitude of the first neglected term. We have proved the following theorem.

Theorem A Alternating Series Test

Let

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(continued on next page)