$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + b_k) + (-1)\sum_{k=1}^{\infty} b_k$, by Theorem B(ii).

41. Taking vertical strips, the area is

$$1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$$

Taking horizontal strips, the area is

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \cdots = \sum_{k=1}^{\infty}\frac{k}{2^k}.$$

a. $\sum_{k=1}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1} = \frac{1}{1-\frac{1}{2}} = 2$

b. The moment about $x = 0$ is

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cdot (1)k = \sum_{k=0}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\frac{k}{2^k} = 2.$$

$$\bar{x} = \frac{\text{moment}}{\text{area}} = \frac{2}{2} = 1$$

43. a. $A = \sum_{n=0}^{\infty} Ce^{-nkt} = \sum_{n=1}^{\infty} C\left(\frac{1}{e^{kt}}\right)^{n-1}$

$$= \frac{C}{1-\frac{1}{e^{kt}}} = \frac{Ce^{kt}}{e^{kt}-1}$$

b. $\frac{1}{2} = e^{-kt} = e^{-6k} \Rightarrow k = \frac{\ln 2}{6} \Rightarrow A = \frac{4}{3}C;$

if $C = 2$ mg, then $A = \frac{8}{3}$ mg.

45. $\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1}f_{k+2}} = \frac{f_{k+2}-f_k}{f_k f_{k+1} f_{k+2}} = \frac{1}{f_k f_{k+2}}$

since $f_{k+2} = f_{k+1} + f_k$. Thus,

$$\sum_{k=1}^{\infty}\frac{1}{f_k f_{k+2}} = \sum_{k=1}^{\infty}\left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1}f_{k+2}}\right)$$ and

$$S_n = \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right) + \left(\frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1}f_{n+2}}\right)$$

$$= \frac{1}{f_1 f_2} - \frac{1}{f_{n+1}f_{n+2}} = \frac{1}{1\cdot 1} - \frac{1}{f_{n+1}f_{n+2}} = 1 - \frac{1}{f_{n+1}f_{n+2}}$$

The terms of the Fibonacci sequence increase without bound, so

$$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{f_{n+1}f_{n+2}} = 1 - 0 = 1$$

10.3 Concepts Review

1. bounded above

3. convergence or divergence

Problem Set 10.3

1. $\frac{1}{x+3}$ is continuous, positive, and nonincreasing on $[0,\infty)$.

$$\int_0^{\infty}\frac{1}{x+3}dx = \left[\ln|x+3|\right]_0^{\infty} = \infty - \ln 3 = \infty$$

The series diverges.

3. $\frac{x}{x^2+3}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^{\infty}\frac{x}{x^2+3}dx = \left[\frac{1}{2}\ln\left|x^2+3\right|\right]_1^{\infty} = \infty - \frac{1}{2}\ln 4 = \infty$$

The series diverges.

5. $\frac{2}{\sqrt{x+2}}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^{\infty}\frac{2}{\sqrt{x+2}}dx = \left[4\sqrt{x+2}\right]_1^{\infty} = \infty - 4\sqrt{3} = \infty$$

Thus $\sum_{k=1}^{\infty}\frac{2}{\sqrt{k+2}}$ diverges, hence

$\sum_{k=1}^{\infty}\frac{-2}{\sqrt{k+2}} = -\sum_{k=1}^{\infty}\frac{2}{\sqrt{k+2}}$ also diverges.

7. $\frac{7}{4x+2}$ is continuous, positive, and nonincreasing on $[2, \infty)$

$$\int_2^{\infty} \frac{7}{4x+2} dx = \left[\frac{7}{4}\ln|4x+2|\right]_2^{\infty} = \infty - \frac{7}{4}\ln 10 = \infty$$

The series diverges.

9. $\frac{3}{(4+3x)^{7/6}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{3}{(4+3x)^{7/6}} dx = \left[-\frac{6}{(4+3x)^{1/6}}\right]_1^{\infty}$$

$$= 0 + \frac{6}{7^{1/6}} = 6 \cdot 7^{-1/6} < \infty$$

The series converges.

11. xe^{-3x^2} is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} xe^{-3x^2} dx = \left[-\frac{1}{6}e^{-3x^2}\right]_1^{\infty} = 0 + \frac{1}{6}e^{-3}$$

$$= \frac{1}{6e^3} < \infty$$

The series converges.

13. $\lim_{k\to\infty} \frac{k^2+1}{k^2+5} = \lim_{k\to\infty} \frac{1+\frac{1}{k^2}}{1+\frac{5}{k^2}} = 1 \neq 0$, so the series diverges.

15. $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ is a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1$ so the series converges.

In $\sum_{k=1}^{\infty} \frac{k-1}{2k+1}$, $\lim_{k\to\infty} \frac{k-1}{2k+1} = \lim_{k\to\infty} \frac{1-\frac{1}{k}}{2+\frac{1}{k}} = \frac{1}{2} \neq 0$, so the series diverges. Thus, the sum of the series diverges.

17. $\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1 & k = 4j+1 \\ -1 & k = 4j+3, \\ 0 & k \text{ is even} \end{cases}$

where j is any nonnegative integer.

Thus $\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right|$ does not exist, hence

$\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right| \neq 0$ and the series diverges.

19. $x^2e^{-x^3}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} x^2e^{-x^3} dx = \left[-\frac{1}{3}e^{-x^3}\right]_1^{\infty} = 0 + \frac{1}{3}e^{-1} < \infty, \text{ so}$$

the series converges.

21. $\frac{\tan^{-1} x}{1+x^2}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx = \left[\frac{1}{2}(\tan^{-1} x)^2\right]_1^{\infty}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32} < \infty, \text{ so the series}$$

converges.

23. $\frac{x}{e^x}$ is continuous, positive, and nonincreasing on $[5, \infty)$.

$$E = \sum_{k=6}^{\infty} \frac{k}{e^k} \le \int_5^{\infty} \frac{x}{e^x} dx = [-xe^{-x}]_5^{\infty} + \int_5^{\infty} e^{-x} dx$$

$$= [-xe^{-x} - e^{-x}]_5^{\infty} = 0 + 5e^{-5} + e^{-5} = 6e^{-5}$$

$$\approx 0.0404$$

25. $\frac{1}{1+x^2}$ is continuous, positive, and nonincreasing on $[5, \infty)$.

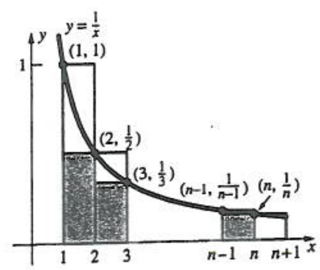
$$E = \sum_{k=6}^{\infty} \frac{1}{1+k^2} \le \int_5^{\infty} \frac{1}{1+x^2} dx = [\tan^{-1} x]_5^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} 5 \approx 0.1974$$

27. Consider $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$. Let $u = \ln x$, $du = \frac{1}{x} dx$.

$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du$ which converges for $p > 1$.

29.



The upper rectangles, which extend to $n + 1$ on the right, have area $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$. These rectangles are above the curve $y=\frac{1}{x}$ from $x = 1$ to $x = n + 1$. Thus,

$$\int_1^{n+1}\frac{1}{x}dx=[\ln x]_1^{n+1}=\ln(n+1)-\ln 1=\ln(n+1)$$

$$<1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

The lower (shaded) rectangles have area $\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$. These rectangles lie below the curve $y=\frac{1}{x}$ from $x = 1$ to $x = n$. Thus

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<\int_1^n\frac{1}{x}dx=\ln n, \text{ so}$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<1+\ln n.$$

31. $\{B_n\}$ is a nondecreasing sequence that is bounded above, thus by the Monotone Sequence Theorem (Theorem D of Section 10.1), $\lim_{n\to\infty} B_n$ exists.
The rationality of γ is a famous unsolved problem.

33. $\gamma+\ln(n+1)>20\Rightarrow\ln(n+1)>20-\gamma\approx 19.4228$

$$\Rightarrow n+1>e^{19.4228}\approx 272,404,867$$

$$\Rightarrow n>272,404,866$$

35. Every time n is incremented by 1, a positive amount of area is added, thus $\{A_n\}$ is an increasing sequence.
Each curved region has horizontal width 1, and can be moved into the heavily outlined triangle without any overlap. This can be done by shifting the nth shaded region, which goes from $(n, f(n))$ to $(n + 1, f(n + 1))$, as follows:
shift $(n + 1, f(n + 1))$ to $(2, f(2))$ and $(n, f(n))$ to $(1, f(2)–[f(n + 1) – f(n)])$.
The slope of the line forming the bottom of the shaded region between $x = n$ and $x = n + 1$ is

$$\frac{f(n+1)-f(n)}{(n+1)-n}=f(n+1)-f(n)>0$$

since f is increasing. By the Mean Value Theorem, $f(n+1)-f(n)=f'(c)$ for some c in $(n, n + 1)$. Since f is concave down, $n < c < n + 1$ means that $f'(c)<f'(b)$ for all b in $[1, n]$. Thus, the nth shaded region will not overlap any other shaded region when shifted into the heavily outlined triangle. Thus, the area of all of the shaded regions is less than or equal to the area of the heavily outlined triangle, so $\lim_{n\to\infty} A_n$ exists.

10.4 Concepts Review

1. $0\le a_k\le b_k$

3. $\rho<1;\ \rho>1;\ \rho=1$

Problem Set 10.4

1. $a_n=\frac{n}{n^2+2n+3};\ b_n=\frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2+2n+3}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{3}{n^2}}=1;$$

$0<1<\infty$

$\sum_{n=1}^{\infty}b_n$ diverges $\Rightarrow\sum_{n=1}^{\infty}a_n$ diverges.

3. $a_n=\frac{1}{n\sqrt{n+1}}=\frac{1}{\sqrt{n^3+n^2}};\ b_n=\frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+n^2}}=\lim_{n\to\infty}\sqrt{\frac{n^3}{n^3+n^2}}$$

$$=\lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}=1;\ 0<1<\infty$$

$\sum_{n=1}^{\infty}b_n$ converges $\Rightarrow\sum_{n=1}^{\infty}a_n$ converges

5. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{8^{n+1}n!}{(n+1)!8^n}=\lim_{n\to\infty}\frac{8}{n+1}=0<1$

The series converges.

7. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!n^{100}}{(n+1)^{100}n!}=\lim_{n\to\infty}\frac{n^{100}}{(n+1)^{99}}$

$$=\lim_{n\to\infty}\frac{n}{\left(\frac{n+1}{n}\right)^{99}}=\infty \text{ since } \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{99}=1$$

The series diverges.

9. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^3(2n)!}{(2n+2)!n^3}$

$$=\lim_{n\to\infty}\frac{(n+1)^3}{(2n+2)(2n+1)n^3}=\lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{4n^5+6n^4+2n^3}$$