Integral Test.

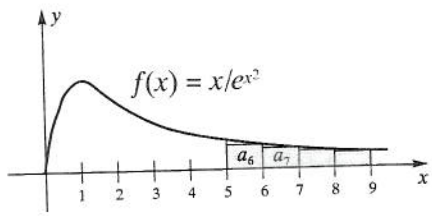

Figure 2

The function $f(x) = x/e^{x^2}$ is continuous, positive, and nonincreasing on $[5, \infty)$ (see Figure 2). Thus,

$$E = \sum_{n=6}^{\infty} \frac{n}{e^{n^2}} < \int_5^{\infty} xe^{-x^2}\,dx$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\right)\int_5^t e^{-x^2}(-2x\,dx)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_5^t = \frac{1}{2}e^{-25} \approx 6.94 \times 10^{-12}$$

■

Concepts Review

1. A series of nonnegative terms converges if and only if its partial sums are ______.

2. The Integral Test relates the convergence of $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$, assuming $a_k =$ ______ and f is ______, ______, and ______ on $[1, \infty)$.

3. The insertion or removal of a finite number of terms in a series does not affect its ______, although it may affect its sum.

4. The p-series $\sum_{k=1}^{\infty}(1/k^p)$ converges if and only if ______.

Problem Set 10.3

Use the Integral Test to decide the convergence or divergence of each of the following series.

1. $\sum_{k=0}^{\infty} \frac{1}{k+3}$

2. $\sum_{k=1}^{\infty} \frac{3}{2k-3}$

3. $\sum_{k=0}^{\infty} \frac{k}{k^2+3}$

4. $\sum_{k=1}^{\infty} \frac{3}{2k^2+1}$

5. $\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}}$

6. $\sum_{k=100}^{\infty} \frac{3}{(k+2)^2}$

7. $\sum_{k=2}^{\infty} \frac{7}{4k+2}$

8. $\sum_{k=1}^{\infty} \frac{k^2}{e^k}$

9. $\sum_{k=1}^{\infty} \frac{3}{(4+3k)^{7/6}}$

10. $\sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3}$

11. $\sum_{k=1}^{\infty} ke^{-3k^2}$

12. $\sum_{k=5}^{\infty} \frac{1000}{k(\ln k)^2}$

use any test you want

In Problems 13–22, use any test developed so far, including any from Section 10.2, to decide about the convergence or divergence of the series. Give a reason for your conclusion.

13. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+5}$

14. $\sum_{k=1}^{\infty} \left(\frac{3}{\pi}\right)^k$

15. $\sum_{k=1}^{\infty} \left[\left(\frac{1}{2}\right)^k + \frac{k-1}{2k+1}\right]$

16. $\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{2^k}\right)$

17. $\sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}\right)$

18. $\sum_{k=1}^{\infty} k \sin\frac{1}{k}$

19. $\sum_{k=1}^{\infty} k^2 e^{-k^3}$

20. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

21. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

22. $\sum_{k=1}^{\infty} \frac{1}{1+4k^2}$

In Problems 23–26, estimate the error that is made by approximating the sum of the given series by the sum of the first five terms (see Example 5).

23. $\sum_{k=1}^{\infty} \frac{k}{e^k}$

24. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$

25. $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

26. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

27. For what values of p does $\sum_{n=2}^{\infty} 1/[n(\ln n)^p]$ converge? Explain.

28. Does $\sum_{n=3}^{\infty} 1/[n \cdot \ln n \cdot \ln(\ln n)]$ converge or diverge? Explain.

29. Use diagrams, as in Figure 1, to show that

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

Hint: $\int_1^n (1/x)\,dx = \ln n$.

30. Using Problem 29, show that the sequence

$$B_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)$$

is increasing and bounded above by 1.

31. Use the result of Problem 29 to prove that $\lim_{n\to\infty} B_n$ exists. (The limit, denoted γ, is called **Euler's constant** and is approximately 0.5772. It is currently not known whether γ is rational or irrational. It is known, however, that *if* γ is rational then the denominator in its reduced fraction is at least $10^{244,663}$.)

32. Use Problem 29 to get good upper and lower bounds for the sum of the first 10 million terms of the harmonic series.

33. From Problem 31, we infer that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \gamma + \ln(n+1)$$

Use this to estimate the number of terms of the harmonic series that are needed to get a sum greater than 20 and compare with the result reported in Problem 38 of Section 10.2.

34. Now that we have shown the existence of Euler's constant the hard way (Problems 29–31), we will solve a much more general problem the easy way and watch γ appear out of thin air, so to speak. Let f be continuous and decreasing on $[1, \infty)$ and let

$$B_n = f(1) + f(2) + \cdots + f(n) - \int_1^{n+1} f(x)\,dx$$

Note that B_n is the area of the shaded region in Figure 3.

(a) Why is it obvious that B_n increases with n?

(b) Show that $B_n \leq f(1)$. *Hint*: Simply shift all the little shaded pieces leftward into the outlined rectangle.

(c) Conclude that $\lim_{n\to\infty} B_n$ exists.

(d) How do we get γ out of this?

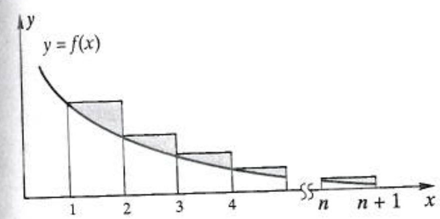

Figure 3

35. Let f be continuous, increasing, and concave down on $[1, \infty)$ as in Figure 4. Furthermore, let A_n be the area of the shaded region. Show that A_n is increasing with n, that $A_n \leq T$ where T is the area of the outlined triangle, and thus that $\lim_{n\to\infty} A_n$ exists.

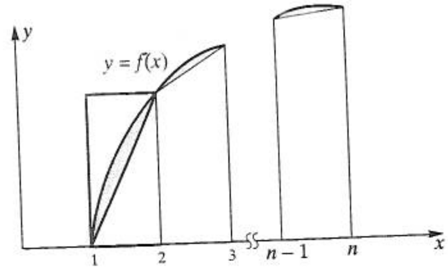

Figure 4

36. Specialize f of Problem 35 to $f(x) = \ln x$.

(a) Show that

$$A_n = \int_1^n \ln x\,dx - \left[\frac{\ln 1 + \ln 2}{2} + \cdots + \frac{\ln(n-1) + \ln n}{2}\right]$$

$$= n \ln n - n + 1 - \ln n! + \ln \sqrt{n}$$

$$= 1 + \ln \frac{(n/e)^n \sqrt{n}}{n!}$$

(b) Conclude from part (a) and Problem 35 that

$$k = \lim_{n\to\infty} \frac{n!}{(n/e)^n \sqrt{n}}$$

exists. It can be shown that $k = \sqrt{2\pi}$.

(c) This means that $n! \approx \sqrt{2\pi n}(n/e)^n$, which is called **Stirling's Formula**. Use it to approximate 15! and compare it with the value that your calculator gives for 15!

Answers to Concepts Review: **1.** bounded above **2.** $f(k)$; continuous; positive; nonincreasing **3.** convergence or divergence **4.** $p > 1$

10.4 Positive Series: Other Tests

We have completely analyzed the convergence and divergence of two series, the geometric series and the p-series.

$$\sum_{n=1}^{\infty} r^n \quad \text{converges if } -1 < r < 1, \text{ diverges otherwise}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p > 1, \text{ diverges otherwise}$$

In the first we have found what the series converges to, provided that it converges; in the second, we have not. These series provide standards, or models, against which we can measure other series. Keep in mind that we are still considering series whose terms are positive (or at least nonnegative).

Comparing One Series with Another A series with terms less than the corresponding terms of a convergent series ought to converge; a series with terms greater than the corresponding terms of a divergent series ought to diverge. What ought to be true is true.