$$= \lim_{x\to 0^+}\left[\left(1+\frac{x}{2}\right)^{2/x}\right]^{1/2} = e^{1/2}, \text{ so}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = e^{1/2}.$$

57. Let $f(x) = \left(\frac{x-1}{x+1}\right)^x$.

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{x\to 0^+}\left(\frac{\frac{1}{x}-1}{\frac{1}{x}+1}\right)^{1/x} = \lim_{x\to 0^+}\left(\frac{\frac{1-x}{x}}{\frac{1+x}{x}}\right)^{1/x}$$

$$= \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}, \text{ so}$$

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = e^{-2}.$$

59. Let $f(x) = \left(\frac{2+x^2}{3+x^2}\right)^{x^2}$.

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^{x^2} = \lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x^2}$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x^2} = \lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2} = e^{-1},$$

so $\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^{n^2} = e^{-1}$.

10.2 Concepts Review

1. an infinite series

3. $|r| < 1; \frac{a}{1-r}$

Problem Set 10.2

1. $\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)^k = \frac{1}{7}+\frac{1}{7}\cdot\frac{1}{7}+\frac{1}{7}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{1}{7}, r=\frac{1}{7}$; $S = \frac{\frac{1}{7}}{1-\frac{1}{7}} = \frac{\frac{1}{7}}{\frac{6}{7}} = \frac{1}{6}$

3. $\sum_{k=0}^{\infty}2\left(\frac{1}{4}\right)^k = 2+2\cdot\frac{1}{4}+2\left(\frac{1}{4}\right)^2+\ldots$; a geometric series with $a=2$, $r=\frac{1}{4}$; $S = \frac{2}{1-\frac{1}{4}} = \frac{2}{\frac{3}{4}} = \frac{8}{3}$.

$\sum_{k=0}^{\infty}3\left(-\frac{1}{5}\right)^k = 3-3\cdot\frac{1}{5}+3\left(\frac{1}{5}\right)^2-\ldots$; a geometric series with $a=3$, $r=-\frac{1}{5}$;

$$S = \frac{3}{1-\left(-\frac{1}{5}\right)} = \frac{3}{\frac{6}{5}} = \frac{5}{2}$$

Thus, by Theorem B,

$$\sum_{k=0}^{\infty}\left[2\left(\frac{1}{4}\right)^k+3\left(-\frac{1}{5}\right)^k\right] = \frac{8}{3}+\frac{5}{2} = \frac{31}{6}$$

5. $\sum_{k=1}^{\infty}\frac{k-5}{k+2} = -\frac{4}{3}-\frac{3}{4}-\frac{2}{5}-\frac{1}{6}+0+\frac{1}{8}+\frac{2}{9}+\ldots$;

$\lim_{k\to\infty}\frac{k-5}{k+2} = \lim_{k\to\infty}\frac{1-\frac{5}{k}}{1+\frac{2}{k}} = 1 \neq 0$; the series diverges.

7. $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right) = \left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\ldots$;

$S_n = \left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n-2}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right) = -1+\frac{1}{n}$;

$\lim_{n\to\infty}S_n = \lim_{n\to\infty}-1+\frac{1}{n} = -1$, so $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right) = -1$

9. $\sum_{k=1}^{\infty}\frac{k!}{100^k} = \frac{1}{100}+\frac{2}{10,000}+\frac{6}{1,000,000}+\ldots$

Consider $\{a_n\}$, where $a_{n+1} = \frac{n+1}{100}a_n, a_1 = \frac{1}{100}$. $a_n > 0$ for all n, and for $n > 99$, $a_{n+1} > a_n$, so the sequence is eventually an increasing sequence, hence $\lim_{n\to\infty} a_n \neq 0$. The sequence can also be described by

$a_n = \frac{n!}{100^n}$, hence $\sum_{k=1}^{\infty} \frac{k!}{100^k}$ diverges.

11. $\sum_{k=1}^{\infty}\left(\frac{e}{\pi}\right)^{k+1} = \left(\frac{e}{\pi}\right)^2 + \left(\frac{e}{\pi}\right)^2 \cdot \frac{e}{\pi} + \left(\frac{e}{\pi}\right)^2\left(\frac{e}{\pi}\right)^2 + \ldots$; a geometric series with $a = \left(\frac{e}{\pi}\right)^2$, $r = \frac{e}{\pi} < 1$;

$$S = \frac{\left(\frac{e}{\pi}\right)^2}{1-\frac{e}{\pi}} = \frac{\left(\frac{e}{\pi}\right)^2}{\frac{\pi-e}{\pi}} = \frac{e^2}{\pi(\pi-e)} \approx 5.5562$$

13. $\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2} - \frac{3}{k^2}\right) = \left(\frac{3}{1} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{3}{9}\right) + \left(\frac{3}{9} - \frac{3}{16}\right) + \ldots;$

$$S_n = \left(3 - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{3}{16}\right) + \ldots + \left(\frac{3}{(n-2)^2} - \frac{3}{(n-1)^2}\right) + \left(\frac{3}{(n-1)^2} - \frac{3}{n^2}\right)$$

$= 3 - \frac{3}{n^2}$; $\lim_{n\to\infty} S_n = 3 - \lim_{n\to\infty} \frac{3}{n^2} = 3$, so

$$\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2} - \frac{3}{k^2}\right) = 3.$$

15. $0.22222\ldots = \sum_{k=1}^{\infty} \frac{2}{10}\left(\frac{1}{10}\right)^{k-1}$

$$= \frac{\frac{2}{10}}{1-\frac{1}{10}} = \frac{2}{9}$$

17. $0.013013013\ldots = \sum_{k=1}^{\infty} \frac{13}{1000}\left(\frac{1}{1000}\right)^{k-1}$

$$= \frac{\frac{13}{1000}}{1-\frac{1}{1000}} = \frac{13}{999}$$

19. $0.4999\ldots = \frac{4}{10} + \sum_{k=1}^{\infty} \frac{9}{100}\left(\frac{1}{10}\right)^{k-1}$

$$= \frac{4}{10} + \frac{\frac{9}{100}}{1-\frac{1}{10}} = \frac{1}{2}$$

21. Let $s = 1 - r$, so $r = 1 - s$. Since $0 < r < 2$, $-1 < 1 - r < 1$, so

$|s| < 1$, and $\sum_{k=0}^{\infty} r(1-r)^k = \sum_{k=0}^{\infty} (1-s)s^k$

$$= \sum_{k=1}^{\infty} (1-s)s^{k-1} = \frac{1-s}{1-s} = 1$$

23. $\ln \frac{k}{k+1} = \ln k - \ln(k+1)$

$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \ldots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$, thus $\sum_{k=1}^{\infty} \ln \frac{k}{k+1}$ diverges.

25. The ball drops 100 feet, rebounds up $100\left(\frac{2}{3}\right)$ feet, drops $100\left(\frac{2}{3}\right)$ feet, rebounds up $100\left(\frac{2}{3}\right)^2$ feet, drops $100\left(\frac{2}{3}\right)^2$, etc. The total distance it travels is

$$100+200\left(\frac{2}{3}\right)+200\left(\frac{2}{3}\right)^2+200\left(\frac{2}{3}\right)^3+\ldots=-100+200+200\left(\frac{2}{3}\right)+200\left(\frac{2}{3}\right)^2+200\left(\frac{2}{3}\right)^3+\ldots$$

$$=-100+\sum_{k=1}^{\infty}200\left(\frac{2}{3}\right)^{k-1}=-100+\frac{200}{1-\frac{2}{3}}=500 \text{ feet}$$

27. $1 billion + 75% of $1 billion + 75% of 75% of $1 billion + ... $=\sum_{k=1}^{\infty}(\$1 \text{ billion})0.75^{k-1}=\frac{\$1\text{ billion}}{1-0.75}=\$4$ billion

29. As the midpoints of the sides of a square are connected, a new square is formed. The new square has sides $\frac{1}{\sqrt{2}}$ times the sides of the old square. Thus, the new square has area $\frac{1}{2}$ the area of the old square. Then in the next step, $\frac{1}{8}$ of each new square is shaded.

$$\text{Area}=\frac{1}{8}\cdot 1+\frac{1}{8}\cdot\frac{1}{2}+\frac{1}{8}\cdot\frac{1}{4}+\ldots=\sum_{k=1}^{\infty}\frac{1}{8}\left(\frac{1}{2}\right)^{k-1}=\frac{\frac{1}{8}}{1-\frac{1}{2}}=\frac{1}{4}$$

The area will be $\frac{1}{4}$.

31. $\frac{3}{4}+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\ldots=\sum_{k=1}^{\infty}\frac{3}{4}\left(\frac{1}{16}\right)^{k-1}=\frac{\frac{3}{4}}{1-\frac{1}{16}}=\frac{4}{5}$

The original does not need to be equilateral since each smaller triangle will have $\frac{1}{4}$ area of the previous larger triangle.

33. Both Achilles and the tortoise will have moved.

$$100+10+1+\frac{1}{10}+\frac{1}{100}+\ldots=\sum_{k=1}^{\infty}100\left(\frac{1}{10}\right)^{k-1}$$

$$=\frac{100}{1-\frac{1}{10}}=111\frac{1}{9} \text{ yards}$$

Also, one can see this by the following reasoning. In the time it takes the tortoise to run $\frac{d}{10}$ yards, Achilles will run d yards. Solve

$$d=100+\frac{d}{10}, d=\frac{1000}{9}=111\frac{1}{9} \text{ yards}$$

35. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}ca_k$ converges, and $c \neq 0$. Then $\frac{1}{c}$ is defined, so $\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}\frac{1}{c}ca_k=\frac{1}{c}\sum_{k=1}^{\infty}ca_k$ would also converge, by Theorem B(i).

37. a. The top block is supported *exactly* at its center of mass. The location of the center of mass of the top n blocks is the average of the locations of their individual centers of mass, so the nth block moves the center of mass left by $\frac{1}{n}$ of the location of its center of mass, that is, $\frac{1}{n}\cdot\frac{1}{2}$ or $\frac{1}{2n}$ to the left. But this is exactly how far the $(n + 1)$st block underneath it is offset.

b. Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$, which diverges, there is no limit to how far the top block can protrude.

39. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}(a_k+b_k)$ converges. Since $\sum_{k=1}^{\infty}b_k$ converges, so would

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + b_k) + (-1)\sum_{k=1}^{\infty} b_k$, by Theorem B(ii).

41. Taking vertical strips, the area is

$$1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$$

Taking horizontal strips, the area is

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \cdots = \sum_{k=1}^{\infty}\frac{k}{2^k}.$$

a. $\sum_{k=1}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1} = \frac{1}{1-\frac{1}{2}} = 2$

b. The moment about $x = 0$ is

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cdot (1)k = \sum_{k=0}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\frac{k}{2^k} = 2.$$

$$\overline{x} = \frac{\text{moment}}{\text{area}} = \frac{2}{2} = 1$$

43. a. $A = \sum_{n=0}^{\infty} Ce^{-nkt} = \sum_{n=1}^{\infty} C\left(\frac{1}{e^{kt}}\right)^{n-1}$

$$= \frac{C}{1-\frac{1}{e^{kt}}} = \frac{Ce^{kt}}{e^{kt}-1}$$

b. $\frac{1}{2} = e^{-kt} = e^{-6k} \Rightarrow k = \frac{\ln 2}{6} \Rightarrow A = \frac{4}{3}C;$

if $C = 2$ mg, then $A = \frac{8}{3}$ mg.

45. $\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1}f_{k+2}} = \frac{f_{k+2} - f_k}{f_k f_{k+1} f_{k+2}} = \frac{1}{f_k f_{k+2}}$

since $f_{k+2} = f_{k+1} + f_k$. Thus,

$$\sum_{k=1}^{\infty}\frac{1}{f_k f_{k+2}} = \sum_{k=1}^{\infty}\left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1}f_{k+2}}\right) \text{ and}$$

$$S_n = \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right) + \left(\frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1}f_{n+2}}\right)$$

$$= \frac{1}{f_1 f_2} - \frac{1}{f_{n+1}f_{n+2}} = \frac{1}{1\cdot 1} - \frac{1}{f_{n+1}f_{n+2}} = 1 - \frac{1}{f_{n+1}f_{n+2}}$$

The terms of the Fibonacci sequence increase without bound, so

$$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{f_{n+1}f_{n+2}} = 1 - 0 = 1$$

10.3 Concepts Review

1. bounded above

3. convergence or divergence

Problem Set 10.3

1. $\frac{1}{x+3}$ is continuous, positive, and nonincreasing on $[0,\infty)$.

$$\int_0^{\infty}\frac{1}{x+3}dx = \left[\ln|x+3|\right]_0^{\infty} = \infty - \ln 3 = \infty$$

The series diverges.

3. $\frac{x}{x^2+3}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^{\infty}\frac{x}{x^2+3}dx = \left[\frac{1}{2}\ln\left|x^2+3\right|\right]_1^{\infty} = \infty - \frac{1}{2}\ln 4 = \infty$$

The series diverges.

5. $\frac{2}{\sqrt{x+2}}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^{\infty}\frac{2}{\sqrt{x+2}}dx = \left[4\sqrt{x+2}\right]_1^{\infty} = \infty - 4\sqrt{3} = \infty$$

Thus $\sum_{k=1}^{\infty}\frac{2}{\sqrt{k+2}}$ diverges, hence

$\sum_{k=1}^{\infty}\frac{-2}{\sqrt{k+2}} = -\sum_{k=1}^{\infty}\frac{2}{\sqrt{k+2}}$ also diverges.