has partial sums

$$
\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 - 1 = 0 \\
S_3 &= 1 - 1 + 1 = 1 \\
S_4 &= 1 - 1 + 1 - 1 = 0 \\
&\vdots
\end{aligned}
$$

• nth term test for divergence
• Geometric Series
• Telescoping Series

The sequence of partial sums, $1, 0, 1, 0, 1, \ldots$, diverges; thus the series $1 - 1 + 1 - 1 + \cdots$ diverges. We might, however, view the series as

$$(1 - 1) + (1 - 1) + \cdots$$

and claim that the sum is 0. Alternatively, we might view the series as

$$1 - (1 - 1) - (1 - 1) - \cdots$$

and claim that the sum is 1. The sum of the series cannot be equal to both 0 and 1. It turns out that grouping of terms in a series is acceptable provided that the series is convergent; in such a case we can group terms in any way that we wish.

Theorem D Grouping Terms in an Infinite Series

The terms of a convergent series can be grouped in any way (provided that the order of the terms is maintained), and the new series will converge with the same sum as the original series.

Proof Let Σa_n be the original convergent series and let $\{S_n\}$ be its sequence of partial sums. If Σb_m is a series formed by grouping the terms of Σa_n and if $\{T_m\}$ is its sequence of partial sums, then each T_m is one of the S_n's. For example, T_4 might be

$$T_4 = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6) + (a_7 + a_8)$$

in which case $T_4 = S_8$. Thus, $\{T_m\}$ is a "subsequence" of $\{S_n\}$. A moment's thought should convince you that if $S_n \to S$ then $T_m \to S$. ◆

Concepts Review

1. An expression of the form $a_1 + a_2 + a_3 + \cdots$ is called ______.

2. A series $a_1 + a_2 + \cdots$ is said to converge if the sequence $\{S_n\}$ converges, where $S_n =$ ______.

3. The geometric series $a + ar + ar^2 + \cdots$ converges if ______; in this case the sum of the series is ______.

4. If $\lim_{n\to\infty} a_n \neq 0$, we can be sure that the series $\sum_{n=1}^{\infty} a_n$ ______.

Problem Set 10.2

In Problems 1–14, indicate whether the given series converges or diverges. If it converges, find its sum. Hint: It may help you to write out the first few terms of the series.

1. $\sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^k$

2. $\sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^{-k-2}$

3. $\sum_{k=0}^{\infty} \left[2\left(\frac{1}{4}\right)^k + 3\left(-\frac{1}{5}\right)^k\right]$

4. $\sum_{k=1}^{\infty} \left[5\left(\frac{1}{2}\right)^k - 3\left(\frac{1}{7}\right)^{k+1}\right]$

5. $\sum_{k=1}^{\infty} \frac{k-5}{k+2}$

6. $\sum_{k=1}^{\infty} \left(\frac{9}{8}\right)^k$

7. $\sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k-1}\right)$ *Hint*: Example 6. → telescoping Series

8. $\sum_{k=1}^{\infty} \frac{3}{k}$

9. $\sum_{k=1}^{\infty} \frac{k!}{100^k}$

10. $\sum_{k=1}^{\infty} \frac{2}{(k+2)k}$

11. $\sum_{k=1}^{\infty} \left(\frac{e}{\pi}\right)^{k+1}$

12. $\sum_{k=1}^{\infty} \frac{4^{k+1}}{7^{k-1}}$

13. $\sum_{k=2}^{\infty} \left(\frac{3}{(k-1)^2} - \frac{3}{k^2}\right)$

14. $\sum_{k=6}^{\infty} \frac{2}{k-5}$

In Problems 15–20, write the given decimal as an infinite series, then find the sum of the series, and finally use the result to write the decimal as a ratio of two integers (see Example 2).

15. 0.22222...

16. 0.21212121...

17. 0.013013013...

18. 0.125125125...

19. 0.49999...

20. 0.36717171...

21. Evaluate $\sum_{k=0}^{\infty} r(1-r)^k, 0 < r < 2$.

22. Evaluate $\sum_{k=0}^{\infty} (-1)^k x^k, -1 < x < 1$.

23. Show that $\sum_{k=1}^{\infty} \ln \frac{k}{k+1}$ diverges. *Hint*: Obtain a formula for S_n.

24. Show that $\sum_{k=2}^{\infty} \ln\left(1 - \frac{1}{k^2}\right) = -\ln 2$.

25. A ball is dropped from a height of 100 feet. Each time it hits the floor, it rebounds to $\frac{2}{3}$ its previous height. Find the total distance it travels before coming to rest.

26. Three people, A, B, and C, divide an apple as follows. First they divide it into fourths, each taking a quarter. Then they divide the leftover quarter into fourths, each taking a quarter, and so on. Show that each gets a third of the apple.

27. Suppose that the government pumps an extra \$1 billion into the economy. Assume that each business and individual saves 25% of its income and spends the rest, so of the initial \$1 billion 75% is respent by individuals and businesses. Of that amount, 75% is spent, and so forth. What is the total increase in spending due to the government action? (This is called the *multiplier effect* in economics.)

28. Do Problem 27 assuming that only 10% of the income is saved at each stage.

29. Assume that square $ABCD$ (Figure 3) has sides of length 1 and that E, F, G, and H are midpoints of the sides. If the indicated pattern is continued indefinitely, what will be the area of the painted region?

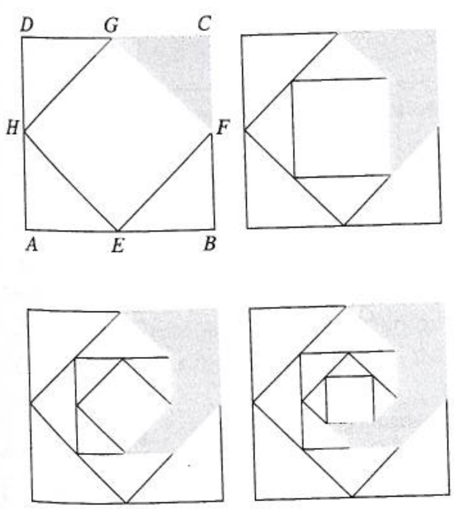

Figure 3

30. If the pattern shown in Figure 4 is continued indefinitely, what fraction of the original square will eventually be painted?

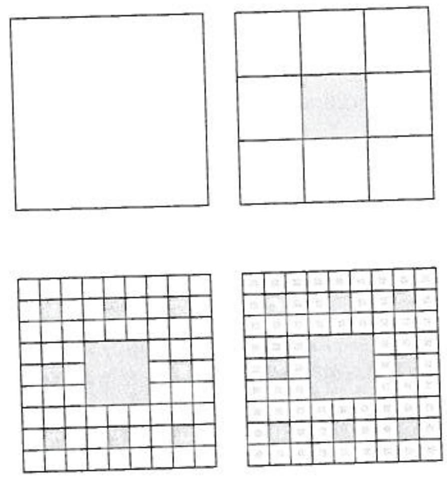

Figure 4

31. Each triangle in the descending chain (Figure 5) has its vertices at the midpoints of the sides of the next larger one. If the indicated pattern of painting is continued indefinitely, what fraction of the original triangle will be painted? Does the original triangle need to be equilateral for this to be true?

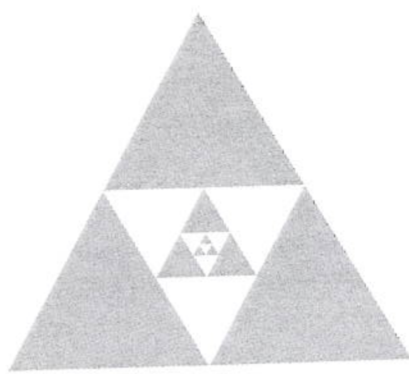

Figure 5

32. Circles are inscribed in the triangles of Problem 31 as indicated in Figure 6. If the original triangle is equilateral, what fraction of the area is eventually painted?

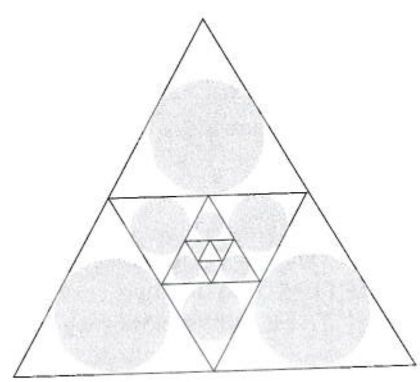

Figure 6

33. In another version of Zeno's paradox, Achilles can run ten times as fast as the tortoise, but the tortoise has a 100–yard headstart. Achilles cannot catch the tortoise, says Zeno, because when Achilles runs 100 yards the tortoise will have moved 10 yards ahead, when Achilles runs another 10 yards, the tortoise will have moved 1 yard ahead, and so on. Convince Zeno that Achilles will catch the tortoise and tell him exactly how many yards Achilles will have to run to do it.

34. Tom and Joel are good runners, both able to run at a constant speed of 10 miles per hour. Their amazing dog Trot can do even better; he runs at 20 miles per hour. Starting from towns 60 miles apart, Tom and Joel run toward each other while Trot runs back and forth between them. How far does Trot run by the time the boys meet? Assume that Trot started with Tom running toward Joel and that he is able to make instant turnarounds. Solve the problem two ways.

(a) Use a geometric series.

(b) Find a shorter way to do the problem.

35. Prove: If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} ca_k$ for $c \neq 0$.

36. Use Problem 35 to conclude that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ diverges.

37. Suppose that one has an unlimited supply of identical blocks each 1 unit long.

(a) Convince yourself that they may be stacked as in Figure 7 without toppling. *Hint*: Consider centers of mass.

(b) How far can one make the top block protrude to the right of the bottom block using this method of stacking?

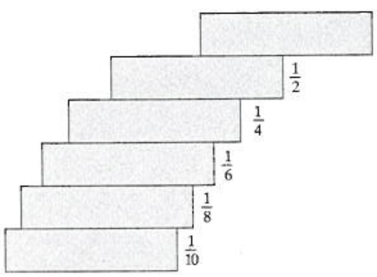

Figure 7

38. How large must N be in order for $S_N = \sum_{k=1}^{N} (1/k)$ just to exceed 4? *Note*: Computer calculations show that for S_N to exceed 20, $N = 272{,}400{,}600$, and for S_N to exceed 100, $N \approx 1.5 \times 10^{43}$.

39. Prove that if Σa_n diverges and Σb_n converges then $\Sigma(a_n + b_n)$ diverges.

40. Show that it is possible for Σa_n and Σb_n both to diverge and yet for $\Sigma(a_n + b_n)$ to converge.

41. By looking at the region in Figure 8 first vertically and then horizontally, conclude that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

and use this fact to calculate:

(a) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

(b) $\bar{x}$, the horizontal coordinate of the centroid of the region.

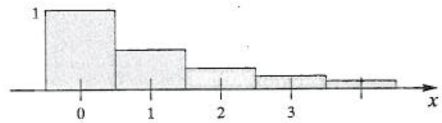

Figure 8

42. Let r be a fixed number with $|r| < 1$. Then it can be shown that $\sum_{k=1}^{\infty} kr^k$ converges, say with sum S. Use the properties of Σ to show that

$$(1 - r)S = \sum_{k=1}^{\infty} r^k$$

and then obtain a formula for S, thus generalizing Problem 41a.

43. Many drugs are eliminated from the body in an exponential manner. Thus, if a drug is given in dosages of size C at time intervals of length t, the amount A_n of the drug in the body just after the $(n + 1)$st dose is

$$A_n = C + Ce^{-kt} + Ce^{-2kt} + \cdots + Ce^{-nkt}$$

where k is a positive constant that depends on the type of drug.

(a) Derive a formula for A, the amount of drug in the body just after a dose if a person has been on the drug for a very long time (assume an infinitely long time).

(b) Evaluate A if it is known that one-half of a dose is eliminated from the body in 6 hours and doses of size 2 milligrams are given every 12 hours.

44. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{2^k}{(2^{k+1} - 1)(2^k - 1)}$$

45. Evaluate $\sum_{k=1}^{\infty} \frac{1}{f_k f_{k+2}}$ where $\{f_k\}$ is the Fibonacci sequence introduced in Problem 52 of Section 10.1. *Hint*: First show that

$$\frac{1}{f_k f_{k+2}} = \frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}}$$

Answers to Concepts Review: 1. an infinite series 2. $a_1 + a_2 + a_3 + \cdots + a_n$ 3. $|r| < 1$; $a/(1 - r)$ 4. diverges

10.3 Positive Series: The Integral Test

We introduced some important ideas in Section 10.2, but we illustrated them mainly for two very special types of series: geometric series and collapsing series. For these series we can give exact formulas for the partial sums S_n, something that we can rarely do for most other types of series. Our task now is to begin a study of very general infinite series.

There are always two important questions to ask about a series.

1. Does the series converge?
2. If it converges, what is its sum?

How shall we answer these questions? Someone may suggest that we use a computer. To answer the first question, simply add up more and more terms of the se-