CHAPTER

10 Infinite Series

10.1 Concepts Review

1. a sequence

3. bounded above

Problem Set 10.1

1. $a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_3 = \frac{3}{8}, a_4 = \frac{4}{11}, a_5 = \frac{5}{14}$

$\lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3}$; converges

3. $a_1 = \frac{6}{3} = 2, a_2 = \frac{18}{9} = 2, a_3 = \frac{38}{17},$

$a_4 = \frac{66}{27} = \frac{22}{9}, a_5 = \frac{102}{39} = \frac{34}{13}$

$\lim_{n\to\infty} \frac{4n^2+2}{n^2+3n-1} = \lim_{n\to\infty} \frac{4+\frac{2}{n^2}}{1+\frac{3}{n}-\frac{1}{n^2}} = 4$; converges

5. $a_1 = \frac{7}{8}, a_2 = \frac{26}{27}, a_3 = \frac{63}{64}, \ a_4 = \frac{124}{125}, a_5 = \frac{215}{216}$

$\lim_{n\to\infty} \frac{n^3+3n^2+3n}{(n+1)^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n}{n^3+3n^2+3n+1}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} = 1$

7. $a_1 = -\frac{1}{3}, a_2 = \frac{2}{4} = \frac{1}{2}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{6} = \frac{2}{3},$

$a_5 = -\frac{5}{7}$

$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1,$ but since it alternates between positive and negative, the sequence diverges.

9. $a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}, a_4 = \frac{1}{4}, a_5 = -\frac{1}{5}$

$-1 \le \cos(n\pi) \le 1$ for all n, so

$-\frac{1}{n} \le \frac{\cos(n\pi)}{n} \le \frac{1}{n}.$

$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$ so by the Squeeze Theorem, the sequence converges to 0.

11. $a_1 = \frac{e^2}{3} \approx 2.4630, a_2 = \frac{e^4}{9} \approx 6.0665,$

$a_3 = \frac{e^6}{17} \approx 23.7311, a_4 = \frac{e^8}{27} \approx 110.4059,$

$a_5 = \frac{e^{10}}{39} \approx 564.7812$

Consider

$\lim_{x\to\infty} \frac{e^{2x}}{x^2+3x-1} = \lim_{x\to\infty} \frac{2e^{2x}}{2x+3} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$

by using l'Hôpital's Rule twice. The sequence diverges.

13. $a_1 = -\frac{\pi}{5} \approx -0.6283, a_2 = \frac{\pi^2}{25} \approx 0.3948,$

$a_3 = -\frac{\pi^3}{125} \approx -0.2481, a_4 = \frac{\pi^4}{625} \approx 0.1559,$

$a_5 = -\frac{\pi^5}{3125} \approx -0.0979$

$\frac{(-\pi)^n}{5^n} = \left(-\frac{\pi}{5}\right)^n, -1 < -\frac{\pi}{5} < 1,$ thus the sequence converges to 0.

15. $a_1 = 2.99, a_2 = 2.9801, a_3 \approx 2.9703,$

$a_4 \approx 2.9606, a_5 \approx 2.9510$

$(0.99)^n$ converges to 0 since $-1 < 0.99 < 1$, thus $2+(0.99)^n$ converges to 2.

17. $a_1 = \frac{\ln 1}{\sqrt{1}} = 0, a_2 = \frac{\ln 2}{\sqrt{2}} \approx 0.4901,$

$a_3 = \frac{\ln 3}{\sqrt{3}} \approx 0.6343, a_4 = \frac{\ln 4}{2} \approx 0.6931,$

$a_5 = \frac{\ln 5}{\sqrt{5}} \approx 0.7198$

Consider $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$ by using l'Hôpital's Rule. Thus, $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} = 0$; converges.

19. $a_1 = \left(1+\frac{2}{1}\right)^{1/2} = \sqrt{3} \approx 1.7321,$

$a_2 = \left(1+\frac{2}{2}\right)^{2/2} = 2,$

$a_3 = \left(1+\frac{2}{3}\right)^{3/2} = \left(\frac{5}{3}\right)^{3/2} \approx 2.1517,$

$a_4 = \left(1+\frac{2}{4}\right)^{4/2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$

$a_5 = \left(1+\frac{2}{5}\right)^{5/2} = \left(\frac{7}{5}\right)^{5/2} \approx 2.3191$

Let $\frac{2}{n} = h$, then as $n \to \infty, h \to 0$ and

$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n/2} = \lim_{h\to 0}(1+h)^{1/h} = e$ by Theorem 7.5A; converges

21. $a_n = \frac{n}{n+1}$ or $a_n = 1-\frac{1}{n+1}$;

$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1$; converges

23. $a_n = (-1)^n\frac{n}{2n-1}$; $\lim_{n\to\infty}\frac{n}{2n-1}$

$= \lim_{n\to\infty}\frac{1}{2-\frac{1}{n}} = \frac{1}{2}$, but due to $(-1)^n$, the terms of the sequence alternate between positive and negative, so the sequence diverges.

25. $a_n = \frac{n}{n^2-(n-1)^2} = \frac{n}{n^2-(n^2-2n+1)} = \frac{n}{2n-1}$;

$\lim_{n\to\infty}\frac{n}{2n-1} = \lim_{n\to\infty}\frac{1}{2-\frac{1}{n}} = \frac{1}{2}$; converges

27. $a_n = n\sin\frac{1}{n}$; $\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$ since

$\lim_{x\to 0}\frac{\sin x}{x} = 1$; converges

29. $a_n = \frac{2^n}{n^2}$;

$\lim_{n\to\infty}\frac{2^n}{n^2} = \lim_{n\to\infty}\frac{2^n\ln 2}{2n} = \lim_{n\to\infty}\frac{2^n(\ln 2)^2}{2} = \infty$; diverges

31. $a_1 = \frac{1}{2}, a_2 = \frac{5}{4}, a_3 = \frac{9}{8}, a_4 = \frac{13}{16}$

a_n is positive for all n, and $a_{n+1} < a_n$ for all $n \geq 2$ since $a_{n+1} - a_n = -\frac{4n-7}{2^{n+1}}$, so $\{a_n\}$ converges to a limit $L \geq 0$.

33. $a_2 = \frac{3}{4}$; $a_3 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right) = \frac{2}{3}$;

$a_4 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right) = \frac{5}{8}$;

$a_5 = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)\left(\frac{24}{25}\right) = \frac{3}{5}$

$a_n > 0$ for all n and $a_{n+1} < a_n$ since

$a_{n+1} = a_n\left(1-\frac{1}{(n+1)^2}\right)$ and $1-\frac{1}{(n+1)^2} < 1$, so $\{a_n\}$ converges to a limit $L \geq 0$.

35. $a_1 = 1, a_2 = 1+\frac{1}{2}(1) = \frac{3}{2}, a_3 = 1+\frac{1}{2}\left(\frac{3}{2}\right) = \frac{7}{4},$

$a_4 = 1+\frac{1}{2}\left(\frac{7}{4}\right) = \frac{15}{8}$

Suppose that $1 < a_n < 2$, then $\frac{1}{2} < \frac{1}{2}a_n < 1$, so $\frac{3}{2} < 1+\frac{1}{2}a_n < 2$, or $\frac{3}{2} < a_{n+1} < 2$. Thus, since $1 < a_2 < 2$, every subsequent term is between $\frac{3}{2}$ and 2.

$a_n < 2$ thus $\frac{1}{2}a_n < 1$, so $a_n < 1+\frac{1}{2}a_n = a_{n+1}$ and the sequence is nondecreasing, so $\{a_n\}$ converges to a limit $L \leq 2$.

37.

n	u_n
1	1.73205
2	2.17533
3	2.27493
4	2.29672
5	2.30146
6	2.30249
7	2.30271
8	2.30276

9	2.30277
10	2.30278
11	2.30278

$\lim_{n\to\infty} u_n \approx 2.3028$

39. If $u = \lim_{n\to\infty} u_n$, then $u = \sqrt{3+u}$ or $u^2 = 3+u$;

$u^2 - u - 3 = 0$ when $u = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$ so

$u = \frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.3028$ since $u > 0$ and

$\frac{1}{2}\left(1-\sqrt{13}\right) < 0.$

41.

n	u_n
1	0
2	1
3	1.1
4	1.11053
5	1.11165
6	1.11177
7	1.11178
8	1.11178

$\lim_{n\to\infty} u_n \approx 1.1118$

43. As $n \to \infty, \frac{k}{n} \to 0$; using $\Delta x = \frac{1}{n}$, an equivalent definite integral is

$\int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1$

≈ 0.4597

45. $\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1};$

$\frac{1}{n+1} < \varepsilon$ is the same as $\frac{1}{\varepsilon} < n+1$. For whatever

ε is given, choose $N > \frac{1}{\varepsilon} - 1$ then

$n \ge N \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon.$

47. Recall from Section 1.2 that every rational number can be written as either a terminating or a repeating decimal.

Thus if the sequence 1, 1.4, 1.41, 1.414, … has a limit within the rational numbers, the terms of the sequence would eventually either repeat or terminate, which they do not since they are the decimal approximations to $\sqrt{2}$, which is irrational. Within the real numbers, the least upper bound is $\sqrt{2}$.

49. If $\{b_n\}$ is bounded, there are numbers N and M with $N \le |b_n| \le M$ for all n. Then

$|a_n N| \le |a_n b_n| \le |a_n M|$.

$\lim_{n\to\infty} |a_n N| = |N| \lim_{n\to\infty} |a_n| = 0$ and

$\lim_{n\to\infty} |a_n M| = |M| \lim_{n\to\infty} |a_n| = 0$, so $\lim_{n\to\infty} |a_n b_n| = 0$

by the Squeeze Theorem, and by Theorem C,

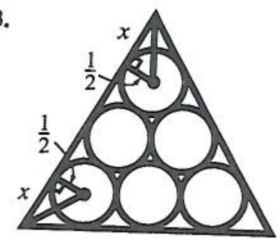
$\lim_{n\to\infty} a_n b_n = 0.$

51. No. Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Both $\{a_n\}$ and $\{b_n\}$ diverge, but

$a_n + b_n = (-1)^n + (-1)^{n+1} = (-1)^n(1+(-1)) = 0$ so

$\{a_n + b_n\}$ converges.

53.



From the figure shown, the sides of the triangle have length $n - 1 + 2x$. The small right triangles marked are 30-60-90 right triangles, so $x = \frac{\sqrt{3}}{2}$; thus the sides of the large triangle have lengths

$n-1+\sqrt{3}$ and $B_n = \frac{\sqrt{3}}{4}\left(n-1+\sqrt{3}\right)^2$

$= \frac{\sqrt{3}}{4}\left(n^2 + 2\sqrt{3}n - 2n - 2\sqrt{3} + 4\right)$ while

$A_n = \frac{n(n+1)}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}(n^2+n)$

$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{\frac{\pi}{8}(n^2+n)}{\frac{\sqrt{3}}{4}(n^2 + 2\sqrt{3}n - 2n - 2\sqrt{3} + 4)}$

$= \lim_{n\to\infty} \frac{\pi\left(1+\frac{1}{n}\right)}{2\sqrt{3}\left(1 + \frac{2\sqrt{3}}{n} - \frac{2}{n} - \frac{2\sqrt{3}}{n^2} + \frac{4}{n^2}\right)} = \frac{\pi}{2\sqrt{3}}$

55. Let $f(x) = \left(1 + \frac{1}{2x}\right)^x$.

$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x = \lim_{x\to 0^+}\left(1+\frac{x}{2}\right)^{1/x}$

$$= \lim_{x\to 0^+}\left[\left(1+\frac{x}{2}\right)^{2/x}\right]^{1/2} = e^{1/2}, \text{ so}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = e^{1/2}.$$

57. Let $f(x)=\left(\frac{x-1}{x+1}\right)^x$.

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{x\to 0^+}\left(\frac{\frac{1}{x}-1}{\frac{1}{x}+1}\right)^{1/x} = \lim_{x\to 0^+}\left(\frac{\frac{1-x}{x}}{\frac{1+x}{x}}\right)^{1/x}$$

$$= \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}, \text{ so}$$

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = e^{-2}.$$

59. Let $f(x)=\left(\frac{2+x^2}{3+x^2}\right)^{x^2}$.

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^{x^2} = \lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x^2}$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x^2} = \lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2} = e^{-1},$$

so $\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^{n^2} = e^{-1}$.

10.2 Concepts Review

1. an infinite series

3. $|r|<1; \frac{a}{1-r}$

Problem Set 10.2

1. $\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)^k = \frac{1}{7}+\frac{1}{7}\cdot\frac{1}{7}+\frac{1}{7}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{1}{7}, r=\frac{1}{7}$; $S=\frac{\frac{1}{7}}{1-\frac{1}{7}}=\frac{\frac{1}{7}}{\frac{6}{7}}=\frac{1}{6}$

3. $\sum_{k=0}^{\infty}2\left(\frac{1}{4}\right)^k = 2+2\cdot\frac{1}{4}+2\left(\frac{1}{4}\right)^2+\ldots$; a geometric series with $a=2$, $r=\frac{1}{4}$; $S=\frac{2}{1-\frac{1}{4}}=\frac{2}{\frac{3}{4}}=\frac{8}{3}$.

$\sum_{k=0}^{\infty}3\left(-\frac{1}{5}\right)^k = 3-3\cdot\frac{1}{5}+3\left(\frac{1}{5}\right)^2-\ldots$; a geometric series with $a=3$, $r=-\frac{1}{5}$;

$$S=\frac{3}{1-\left(-\frac{1}{5}\right)}=\frac{3}{\frac{6}{5}}=\frac{5}{2}$$

Thus, by Theorem B,

$$\sum_{k=0}^{\infty}\left[2\left(\frac{1}{4}\right)^k+3\left(-\frac{1}{5}\right)^k\right]=\frac{8}{3}+\frac{5}{2}=\frac{31}{6}$$

5. $\sum_{k=1}^{\infty}\frac{k-5}{k+2}=-\frac{4}{3}-\frac{3}{4}-\frac{2}{5}-\frac{1}{6}+0+\frac{1}{8}+\frac{2}{9}+\ldots$;

$\lim_{k\to\infty}\frac{k-5}{k+2}=\lim_{k\to\infty}\frac{1-\frac{5}{k}}{1+\frac{2}{k}}=1\neq 0$; the series diverges.

7. $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\ldots$;

$S_n=\left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n-2}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right)=-1+\frac{1}{n}$;

$\lim_{n\to\infty}S_n=\lim_{n\to\infty}-1+\frac{1}{n}=-1$, so $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=-1$

9. $\sum_{k=1}^{\infty}\frac{k!}{100^k}=\frac{1}{100}+\frac{2}{10{,}000}+\frac{6}{1{,}000{,}000}+\ldots$

Consider $\{a_n\}$, where $a_{n+1}=\frac{n+1}{100}a_n, a_1=\frac{1}{100}$. $a_n>0$ for all n, and for $n>99$, $a_{n+1}>a_n$, so the