 Infinite Sequences

## Concepts Review

1. An ~~arrangement~~ ordered list of numbers $a_1, a_2, a_3, \dots$ is called ________.

2. We say the sequence $\{a_n\}$ converges if ________.

3. An increasing sequence that is also ________ must converge.

4. The sequence $\{r^n\}$ converges if and only if ________ $<$ $r \leq$ ________.

## Problem Set 10.1

*In Problems 1–20, an explicit formula for $a_n$ is given. Write the first five terms of $\{a_n\}$, determine whether the sequence converges or diverges, and, if it converges, find $\lim_{n\to\infty} a_n$.*

**1.** $a_n = \dfrac{n}{3n-1}$

**2.** $a_n = \dfrac{3n+2}{n+1}$

**3.** $a_n = \dfrac{4n^2+2}{n^2+3n-1}$

**4.** $a_n = \dfrac{3n^2+2}{2n-1}$

**5.** $a_n = \dfrac{n^3+3n^2+3n}{(n+1)^3}$

**6.** $a_n = \dfrac{\sqrt{3n^2+2}}{2n+1}$

**7.** $a_n = (-1)^n \dfrac{n}{n+2}$

**8.** $a_n = \dfrac{n\cos(n\pi)}{2n-1}$

**9.** $a_n = \dfrac{\cos(n\pi)}{n}$

**10.** $a_n = e^{-n}\sin n$

**11.** $a_n = \dfrac{e^{2n}}{n^2+3n-1}$

**12.** $a_n = \dfrac{e^{2n}}{4^n}$

**13.** $a_n = \dfrac{(-\pi)^n}{5^n}$

**14.** $a_n = \left(\frac{1}{4}\right)^n + 3^{n/2}$

**15.** $a_n = 2 + (0.99)^n$

**16.** $a_n = \dfrac{n^{100}}{e^n}$

**17.** $a_n = \dfrac{\ln n}{\sqrt{n}}$

**18.** $a_n = \dfrac{\ln(1/n)}{\sqrt{2n}}$

**19.** $a_n = \left(1 + \dfrac{2}{n}\right)^{n/2}$

**20.** $a_n = (2n)^{1/2n}$

*Hint:* Theorem 7.5A. = L' Hopital's rule

*In Problems 21–30, find an explicit formula $a_n =$ ________ for each sequence, determine whether the sequence converges or diverges, and, if it converges, find $\lim_{n\to\infty} a_n$.*

**21.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

**22.** $\dfrac{1}{2^2}, \dfrac{2}{2^3}, \dfrac{3}{2^4}, \dfrac{4}{2^5}, \dots$

**23.** $-1, \frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \dots$

**24.** $1, \dfrac{1}{1-\frac{1}{2}}, \dfrac{1}{1-\frac{2}{3}}, \dfrac{1}{1-\frac{3}{4}}, \dots$

**25.** $1, \dfrac{2}{2^2-1^2}, \dfrac{3}{3^2-2^2}, \dfrac{4}{4^2-3^2}, \dots$

**26.** $\dfrac{1}{2-\frac{1}{2}}, \dfrac{2}{3-\frac{1}{3}}, \dfrac{3}{4-\frac{1}{4}}, \dfrac{4}{5-\frac{1}{5}}, \dots$

**27.** $\sin 1, 2\sin\frac{1}{2}, 3\sin\frac{1}{3}, 4\sin\frac{1}{4}, \dots$

**28.** $-\frac{1}{3}, \frac{4}{9}, -\frac{9}{27}, \frac{16}{81}, \dots$

**29.** $2, 1, \dfrac{2^3}{3^2}, \dfrac{2^4}{4^2}, \dfrac{2^5}{5^2}, \dots$

**30.** $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \dots$

*In Problems 31–36, write the first four terms of the sequence $\{a_n\}$. Then use Theorem D to show that the sequence converges.*

**31.** $a_n = \dfrac{4n-3}{2^n}$

**32.** $a_n = \dfrac{n}{n+1}\left(2 - \dfrac{1}{n^2}\right)$

**33.** $a_n = \left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\cdots\left(1 - \dfrac{1}{n^2}\right), n \geq 2$

**34.** $a_n = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

**35.** $a_1 = 1, a_{n+1} = 1 + \frac{1}{2}a_n$

**36.** $a_1 = 2, a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$

$\boxed{C}$ **37.** Assuming that $u_1 = \sqrt{3}$ and $u_{n+1} = \sqrt{3 + u_n}$ determine a convergent sequence, find $\lim_{n\to\infty} u_n$ to four decimal places.

**38.** Show that $\{u_n\}$ of Problem 37 is bounded above and increasing. Conclude from Theorem D that $\{u_n\}$ converges. *Hint:* Use mathematical induction.

**39.** Find $\lim_{n\to\infty} u_n$ of Problem 37 algebraically. *Hint:* Let $u = \lim_{n\to\infty} u_n$. Then, since $u_{n+1} = \sqrt{3 + u_n}$, $u = \sqrt{3 + u}$. Now square both sides and solve for $u$.

**40.** Use the technique of Problem 39 to find $\lim_{n\to\infty} a_n$ of Problem 36.

$\boxed{C}$ **41.** Assuming that $u_1 = 0$ and $u_{n+1} = 1.1^{u_n}$ determine a convergent sequence, find $\lim_{n\to\infty} u_n$ to four decimal places.

**42.** Show that $\{u_n\}$ of Problem 41 is increasing and bounded above by 2.

**43.** Find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sin\frac{k}{n}\right)\frac{1}{n}$$

*Hint:* Write an equivalent definite integral.

**44.** Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{1 + (k/n)^2}\right]\frac{1}{n} = \frac{\pi}{4}$$

**45.** Using the definition of limit, prove that $\lim_{n\to\infty} n/(n+1) = 1$; that is, for a given $\varepsilon > 0$, find $N$ such that $n \geq N \Rightarrow |n/(n+1) - 1| < \varepsilon$.

**46.** As in Problem 45, prove that $\lim_{n\to\infty} n/(n^2+1) = 0$.

**47.** Let $S = \{x : x \text{ is rational and } x^2 < 2\}$. Convince yourself that $S$ does not have a *least* upper bound in the rational numbers, but does have such a bound in the real numbers. In other words, the sequence of rational numbers 1, 1.4, 1.41, 1.414, ..., has no limit within the rational numbers.

EXPL **48.** The *completeness property* of the real numbers says that for every set of real numbers that is bounded above there exists a real number that is a least upper bound for the set. This property is usually taken as an axiom for the real numbers. Prove Theorem D using this property.

**49.** Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded then $\lim_{n\to\infty} a_n b_n = 0$.

**50.** Prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges then $\{a_n + b_n\}$ diverges.

**51.** If $\{a_n\}$ and $\{b_n\}$ both diverge, does it follow that $\{a_n + b_n\}$ diverges?

EXPL **52.** A famous sequence $\{f_n\}$, called the **Fibonacci Sequence** after Leonardo Fibonacci, who introduced it around A.D. 1200, is defined by the recursion formula

$$f_1 = f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n$$

(a) Find $f_3$ through $f_{10}$.

(b) Let $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618034$. The Greeks called this number the *golden ratio*, claiming that a rectangle whose dimensions were in this ratio was "perfect." It can be shown that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
$$= \frac{1}{\sqrt{5}}\left[\phi^n - (-1)^n \phi^{-n}\right]$$

Check that this gives the right result for $n = 1$ and $n = 2$. The general result can be proved by induction (it is a nice challenge). More in line with this section, use this explicit formula to prove that $\lim_{n\to\infty} f_{n+1}/f_n = \phi$.

(c) Using the limit just proved, show that $\phi$ satisfies the equation $x^2 - x - 1 = 0$. Then, in another interesting twist, use the Quadratic Formula to show that the two roots of this equation are $\phi$ and $-1/\phi$, two numbers that occur in the explicit formula for $f_n$.

**53.** Consider an equilateral triangle containing $1 + 2 + 3 + \cdots + n = n(n+1)/2$ circles, each of diameter 1 and stacked as indicated in Figure 4 for the case $n = 4$. Find $\lim_{n\to\infty} A_n/B_n$, where $A_n$ is the total area of the circles and $B_n$ is the area of the triangle.

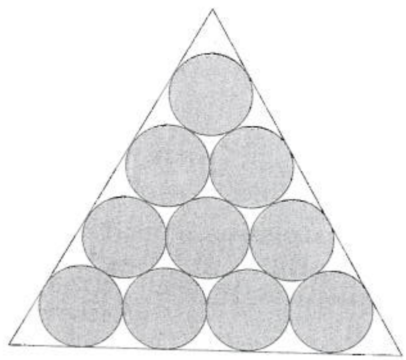

Figure 4

C *Use the fact that* $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$ *to find the following limits.*

**54.** $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

**55.** $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^n$

**56.** $\lim_{n\to\infty}\left(1 + \frac{1}{n^2}\right)^n$

**57.** $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n$

**58.** $\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^n$

**59.** $\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^{n^2}$

**Answers to Concepts Review:** **1.** a sequence **2.** $\lim_{n\to\infty} a_n$ exists (finite sense) **3.** bounded above **4.** $-1$; 1

## 10.2 Infinite Series

In a famous paradox known at least 2400 years ago, Zeno of Elea said that a runner cannot finish a race because he must first cover half the distance, then half the remaining distance, then half the still remaining distance, and so on, forever. Since the runner's time is finite, he cannot traverse the infinite number of segments of the course. Yet we all know that runners do finish races.

Imagine a race course to be 1 mile long. The segments of Zeno's argument would then have lengths $\frac{1}{2}$ mile, $\frac{1}{4}$ mile, $\frac{1}{8}$ mile, and so on (Figure 1). In mathematical language, finishing the race would amount to evaluating the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

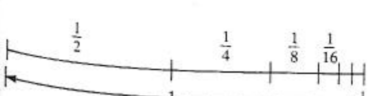


Figure 1

which might seem impossible. But wait. Up to now the word *sum* has been defined only for the addition of a finite number of terms. The indicated "infinite sum" has, as yet, no meaning for us.