35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$
Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$
Hint: Let $u = x^{3/2}$.

Theory and Examples

41. **Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

42. **Volume** Find the volume of the solid generated by revolving the region in Exercise 41 about the x-axis.

43. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

44. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

45. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

46. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

47. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does. Evaluate

$$\int (1+3x^3)e^{x^3}\,dx.$$

48. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1+\sin^2 x}.$$

49. Use the substitution $u = x^4+1$ to evaluate the integral

$$\int x^7\sqrt{x^4+1}\,dx.$$

50. **Using different substitutions** Show that the integral

$$\int ((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x+1)$

b. $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x-1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx,$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du and v, and finally we apply the formula in Equation (2).

EXAMPLE 1 Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

There are four apparent choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x\cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x\cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $du = (\cos x - x \sin x)\,dx$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\,dx.$$

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. When finding v from dv, any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in v because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - \int dx = x \ln x - x + C. \quad \blacksquare$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 3 Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\ &= x^2 e^x - 2x e^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. $\blacksquare$

The technique of Example 3 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x \, dx,$$

so that

$$du = (n-1)\cos^{n-2} x \, (-\sin x \, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx\\ &= \cos^{n-1} x \sin x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x \, dx\\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x \, dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx.$$ ■

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned} \int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x\, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

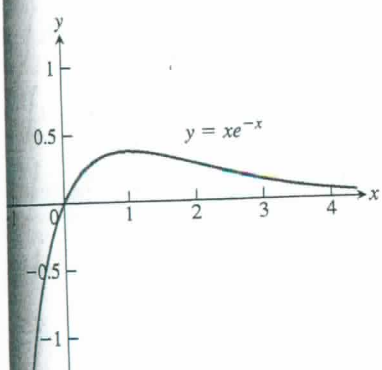

FIGURE 8.1 The region in Example 6.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned} \int_0^4 xe^{-x}\, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \\ &= [-4e^{-4} - (-0e^{-0})] + \int_0^4 e^{-x}\, dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91. \end{aligned}$$ ■

Tabular Integration Can Simplify Repeated Integrations

We have seen that integrals of the form $\int f(x)g(x)\, dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the notation and calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a nice way to organize the calculations that prevents these pitfalls and simplifies the work. It is called **tabular integration** and is illustrated in the next examples.

EXAMPLE 7 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3.

EXAMPLE 8 Find the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution The integral is

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx &= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx \\
&= \frac{1}{n\pi} \sin nx \Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x^3 \cos nx \, dx.
\end{aligned}$$

Using tabular integration to find an antiderivative, we have

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n} \sin nx$
$6x$	$(+)$	$-\frac{1}{n^2} \cos nx$
6	$(-)$	$-\frac{1}{n^3} \sin nx$
0		$\frac{1}{n^4} \cos nx$

the

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{3x^3}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{3\pi^2\cos n\pi}{n^2} - \frac{6\cos n\pi}{n^4} + \frac{6}{n^4}\right)$$

$$= \frac{3}{\pi}\left(\frac{\pi^2 n^2(-1)^n + 2(-1)^{n+1} + 2}{n^4}\right). \qquad \cos n\pi = (-1)^n$$

Integrals like those in Example 8 occur frequently in electrical engineering.

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Exercises 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\,dx$
2. $\int \theta \cos \pi\theta\,d\theta$
3. $\int t^2 \cos t\,dt$
4. $\int x^2 \sin x\,dx$
5. $\int_1^2 x \ln x\,dx$
6. $\int_1^e x^3 \ln x\,dx$
7. $\int xe^x\,dx$
8. $\int xe^{3x}\,dx$
9. $\int x^2e^{-x}\,dx$
10. $\int (x^2 - 2x + 1)e^{2x}\,dx$
11. $\int \tan^{-1} y\,dy$
12. $\int \sin^{-1} y\,dy$
13. $\int x \sec^2 x\,dx$
14. $\int 4x \sec^2 2x\,dx$
15. $\int x^3e^x\,dx$
16. $\int p^4e^{-p}\,dp$
17. $\int (x^2 - 5x)e^x\,dx$
18. $\int (r^2 + r + 1)e^r\,dr$
19. $\int x^5e^x\,dx$
20. $\int t^2e^{4t}\,dt$
21. $\int e^{\theta} \sin \theta\,d\theta$
22. $\int e^{-y} \cos y\,dy$
23. $\int e^{2x} \cos 3x\,dx$
24. $\int e^{-2x} \sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$
26. $\int_0^1 x\sqrt{1-x}\,dx$
27. $\int_0^{\pi/3} x \tan^2 x\,dx$
28. $\int \ln (x + x^2)\,dx$
29. $\int \sin (\ln x)\,dx$
30. $\int z(\ln z)^2\,dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x \sec x^2\,dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx$
33. $\int x (\ln x)^2\,dx$
34. $\int \frac{1}{x(\ln x)^2}\,dx$
35. $\int \frac{\ln x}{x^2}\,dx$
36. $\int \frac{(\ln x)^3}{x}\,dx$
37. $\int x^3 e^{x^4}\,dx$
38. $\int x^5 e^{x^3}\,dx$
39. $\int x^3\sqrt{x^2+1}\,dx$
40. $\int x^2 \sin x^3\,dx$
41. $\int \sin 3x \cos 2x\,dx$
42. $\int \sin 2x \cos 4x\,dx$
43. $\int \sqrt{x} \ln x\,dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
45. $\int \cos \sqrt{x}\,dx$
46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\,d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\,dx$
49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\,dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\,dx$
51. $\int x \tan^{-1} x\,dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\,dx$

Theory and Examples

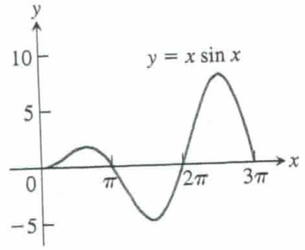
53. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$.

b. $\pi \le x \le 2\pi$.

c. $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n + 1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



54. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for

a. $\pi/2 \le x \le 3\pi/2$.

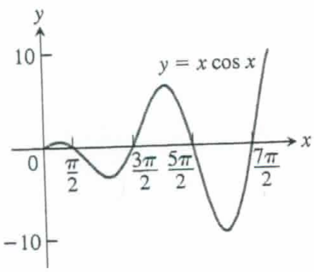
b. $3\pi/2 \le x \le 5\pi/2$.

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



55. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

56. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis.

b. about the line $x = 1$.

57. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis.

b. the line $x = \pi/2$.

58. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

a. the y-axis.

b. the line $x = \pi$.

(See Exercise 53 for a graph.)

59. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

c. Find the volume of the solid formed by revolving this region about the line $x = -2$.

d. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.

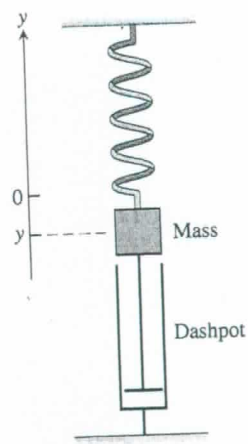
a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the y-axis.

61. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



62. Average value In a mass-spring-dashpot system like the one in Exercise 61, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 63–67, use integration by parts to establish the reduction formula.

63. $\displaystyle\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

64. $\displaystyle\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$

65. $\displaystyle\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

66. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

67. $\displaystyle\int x^m(\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\cdot$

$\displaystyle\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$

68. Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\dfrac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

69. Show that

$$\int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx.$$

70. Use integration by parts to obtain the formula

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\,dx = \int yf'(y)\,dy \qquad y = f^{-1}(x),\quad x = f(y),\quad dx = f'(y)\,dy$$

$$= yf(y) - \int f(y)\,dy \qquad \text{Integration by parts with } u = y,\ dv = f'(y)\,dy$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int ye^y\,dy \qquad y = \ln x,\quad x = e^y,\quad dx = e^y\,dy$$

$$= ye^y - e^y + C$$

$$= x\ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = x\cos^{-1}x - \int\cos y\,dy \qquad y = \cos^{-1}x$$

$$= x\cos^{-1}x - \sin y + C$$

$$= x\cos^{-1}x - \sin(\cos^{-1}x) + C.$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \quad (4)$$

to evaluate the integrals in Exercises 71–74. Express your answers in terms of x.

71. $\displaystyle\int \sin^{-1} x\,dx$

72. $\displaystyle\int \tan^{-1} x\,dx$

73. $\displaystyle\int \sec^{-1} x\,dx$

74. $\displaystyle\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \qquad (5)$$

Exercises 75 and 76 compare the results of using Equations (4) and (5).

75. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\displaystyle\int \cos^{-1}x\,dx = x\cos^{-1}x - \sin(\cos^{-1}x) + C$ Eq. (4)

b. $\displaystyle\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

76. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \ln\sec(\tan^{-1}x) + C$ Eq. (4)

b. $\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 77 and 78 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

77. $\displaystyle\int \sinh^{-1} x\,dx$

78. $\displaystyle\int \tanh^{-1} x\,dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$

Section 8.2, pp. 467–469

1. $-2x\cos(x/2) + 4\sin(x/2) + C$
3. $t^2\sin t + 2t\cos t - 2\sin t + C$
5. $\ln 4 - \frac{3}{4}$ **7.** $xe^x - e^x + C$
9. $-(x^2 + 2x + 2)e^{-x} + C$
11. $y\tan^{-1}(y) - \ln\sqrt{1 + y^2} + C$
13. $x\tan x + \ln|\cos x| + C$
15. $(x^3 - 3x^2 + 6x - 6)e^x + C$ **17.** $(x^2 - 7x + 7)e^x + C$
19. $(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$
21. $\frac{1}{2}(-e^\theta\cos\theta + e^\theta\sin\theta) + C$
23. $\frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$
25. $\frac{2}{3}\left(\sqrt{3s + 9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$
27. $\frac{\pi\sqrt{3}}{3} - \ln(2) - \frac{\pi^2}{18}$
29. $\frac{1}{2}[-x\cos(\ln x) + x\sin(\ln x)] + C$
31. $\frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$
33. $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$
35. $-\frac{1}{x}\ln x - \frac{1}{x} + C$ **37.** $\frac{1}{4}e^{x^4} + C$
39. $\frac{1}{3}x^2(x^2 + 1)^{3/2} - \frac{2}{15}(x^2 + 1)^{5/2} + C$
41. $-\frac{2}{5}\sin 3x\sin 2x - \frac{3}{5}\cos 3x\cos 2x + C$
43. $\frac{2}{9}x^{3/2}(3\ln x - 2) + C$
45. $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$
47. $\frac{\pi^2 - 4}{8}$ **49.** $\frac{5\pi - 3\sqrt{3}}{9}$
51. $\frac{1}{2}(x^2 + 1)\tan^{-1}x - \frac{x}{2} + C$
53. **(a)** π **(b)** 3π **(c)** 5π **(d)** $(2n + 1)\pi$
55. $2\pi(1 - \ln 2)$ **57.** **(a)** $\pi(\pi - 2)$ **(b)** 2π
59. **(a)** 1 **(b)** $(e - 2)\pi$ **(c)** $\frac{\pi}{2}(e^2 + 9)$
(d) $\bar{x} = \frac{1}{4}(e^2 + 1), \bar{y} = \frac{1}{2}(e - 2)$
61. $\frac{1}{2\pi}(1 - e^{-2\pi})$ **63.** $u = x^n, dv = \cos x\,dx$
65. $u = x^n, dv = e^{ax}\,dx$ **71.** $x\sin^{-1}x + \cos(\sin^{-1}x) + C$
73. $x\sec^{-1}x - \ln|x + \sqrt{x^2 - 1}| + C$ **75.** Yes
77. **(a)** $x\sinh^{-1}x - \cosh(\sinh^{-1}x) + C$
(b) $x\sinh^{-1}x - (1 + x^2)^{1/2} + C$

Section 8.3, pp. 474–475

1. $\frac{1}{2}\sin 2x + C$ **3.** $-\frac{1}{4}\cos^4 x + C$
5. $\frac{1}{3}\cos^3 x - \cos x + C$
7. $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$
9. $\sin x - \frac{1}{3}\sin^3 x + C$ **11.** $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$
13. $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$ **15.** $16/35$ **17.** 3π
19. $-4\sin x\cos^3 x + 2\cos x\sin x + 2x + C$
21. $-\cos^4 2\theta + C$ **23.** 4 **25.** 2
27. $\sqrt{\frac{3}{2}} - \frac{2}{3}$ **29.** $\frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{18}{35} - \frac{2}{7}\left(\frac{3}{2}\right)^{7/2}$ **31.** $\sqrt{2}$
33. $\frac{1}{2}\tan^2 x + C$ **35.** $\frac{1}{3}\sec^3 x + C$ **37.** $\frac{1}{3}\tan^3 x + C$
39. $2\sqrt{3} + \ln(2 + \sqrt{3})$ **41.** $\frac{2}{3}\tan\theta + \frac{1}{3}\sec^2\theta\tan\theta + C$
43. $4/3$ **45.** $2\tan^2 x - 2\ln(1 + \tan^2 x) + C$
47. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$ **49.** $\frac{4}{3} - \ln\sqrt{3}$
51. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$ **53.** π
55. $\frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$
57. $\frac{1}{6}\sin 3\theta - \frac{1}{4}\sin\theta - \frac{1}{20}\sin 5\theta + C$
59. $-\frac{2}{5}\cos^5\theta + C$ **61.** $\frac{1}{4}\cos\theta - \frac{1}{20}\cos 5\theta + C$
63. $\sec x - \ln|\csc x + \cot x| + C$ **65.** $\cos x + \sec x + C$
67. $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$ **69.** $\ln(1 + \sqrt{2})$
71. $\pi^2/2$ **73.** $\bar{x} = \frac{4\pi}{3}, \bar{y} = \frac{8\pi^2 + 3}{12\pi}$

Section 8.4, pp. 479–480

1. $\ln|\sqrt{9 + x^2} + x| + C$ **3.** $\pi/4$ **5.** $\pi/6$
7. $\frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25 - t^2}}{2} + C$
9. $\frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7}\right| + C$
11. $7\left[\frac{\sqrt{y^2 - 49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$ **13.** $\frac{\sqrt{x^2 - 1}}{x} + C$
15. $-\sqrt{9 - x^2} + C$ **17.** $\frac{1}{3}(x^2 + 4)^{3/2} - 4\sqrt{x^2 + 4} + C$
19. $\frac{-2\sqrt{4 - w^2}}{w} + C$ **21.** $\sin^{-1}x - \sqrt{1 - x^2} + C$
23. $4\sqrt{3} - \frac{4\pi}{3}$ **25.** $-\frac{x}{\sqrt{x^2 - 1}} + C$
27. $-\frac{1}{5}\left(\frac{\sqrt{1 - x^2}}{x}\right)^5 + C$ **29.** $2\tan^{-1}2x + \frac{4x}{(4x^2 + 1)} + C$
31. $\frac{1}{2}x^2 + \frac{1}{2}\ln|x^2 - 1| + C$ **33.** $\frac{1}{3}\left(\frac{v}{\sqrt{1 - v^2}}\right)^3 + C$
35. $\ln 9 - \ln(1 + \sqrt{10})$ **37.** $\pi/6$ **39.** $\sec^{-1}|x| + C$
41. $\sqrt{x^2 - 1} + C$ **43.** $\frac{1}{2}\ln|\sqrt{1 + x^4} + x^2| + C$
45. $4\sin^{-1}\frac{\sqrt{x}}{2} + \sqrt{x}\sqrt{4 - x} + C$
47. $\frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{4}\sqrt{x}\sqrt{1 - x}(1 - 2x) + C$
49. $y = 2\left[\frac{\sqrt{x^2 - 4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right]$
51. $y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$ **53.** $3\pi/4$