

70. Let  $f(x) = ax^2 + bx + c$ . We calculate the partial fraction decomposition of  $\frac{f(x)}{x^2(x+1)^3}$ . Since  $f(0) = 1$ , we must have

$c = 1$ , so  $\frac{f(x)}{x^2(x+1)^3} = \frac{ax^2 + bx + 1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$ . Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have  $A = C = 0$ , so  $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$ . Equating constant terms gives  $B = 1$ , then equating coefficients of  $x$  gives  $3B = b \Rightarrow b = 3$ . This is the quantity we are looking for, since  $f'(0) = b$ .

## 7.5 Strategy for Integration

1. Let  $u = \sin x$ , so that  $du = \cos x dx$ . Then  $\int \cos x(1 + \sin^2 x) dx = \int (1 + u^2) du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$ .

$$\begin{aligned} 2. \int \frac{\sin^3 x}{\cos x} dx &= \int \frac{\sin^2 x \sin x}{\cos x} dx = \int \frac{(1 - \cos^2 x) \sin x}{\cos x} dx = \int \frac{1 - u^2}{u} (-du) \quad \left[ \begin{array}{l} u = \cos x \\ du = -\sin x dx \end{array} \right] \\ &= \int (u - \frac{1}{u}) du = \frac{1}{2}u^2 - \ln|u| + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C \end{aligned}$$

$$3. \int \frac{\sin x + \sec x}{\tan x} dx = \int \left( \frac{\sin x}{\tan x} + \frac{\sec x}{\tan x} \right) dx = \int (\cos x + \csc x) dx = \sin x + \ln|\csc x - \cot x| + C$$

$$\begin{aligned} 4. \int \tan^3 \theta d\theta &= \int (\sec^2 \theta - 1) \tan \theta d\theta = \int \tan \theta \sec^2 \theta d\theta - \int \frac{\sin \theta}{\cos \theta} d\theta \\ &= \int u du + \int \frac{dv}{v} \quad \left[ \begin{array}{l} u = \tan \theta, \quad v = \cos \theta, \\ du = \sec^2 \theta d\theta \quad dv = -\sin \theta d\theta \end{array} \right] \\ &= \frac{1}{2}u^2 + \ln|v| + C = \frac{1}{2}\tan^2 \theta + \ln|\cos \theta| + C \end{aligned}$$

$$\begin{aligned} 5. \int_0^2 \frac{2t}{(t-3)^2} dt &= \int_{-3}^{-1} \frac{2(u+3)}{u^2} du \quad \left[ \begin{array}{l} u = t-3, \\ du = dt \end{array} \right] = \int_{-3}^{-1} \left( \frac{2}{u} + \frac{6}{u^2} \right) du = \left[ 2\ln|u| - \frac{6}{u} \right]_{-3}^{-1} \\ &= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9 \end{aligned}$$

$$6. \text{Let } u = x^2. \text{ Then } du = 2x dx \Rightarrow \int \frac{x dx}{\sqrt{3-x^4}} = \frac{1}{2} \int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2} \sin^{-1} \frac{u}{\sqrt{3}} + C = \frac{1}{2} \sin^{-1} \frac{x^2}{\sqrt{3}} + C.$$

$$7. \text{Let } u = \arctan y. \text{ Then } du = \frac{dy}{1+y^2} \Rightarrow \int_{-1}^1 \frac{e^{\arctan y}}{1+y^2} dy = \int_{-\pi/4}^{\pi/4} e^u du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$$

$$8. \int x \csc x \cot x dx \quad \left[ \begin{array}{l} u = x, \quad du = dx \\ dv = \csc x \cot x dx, \quad v = -\csc x \end{array} \right] = -x \csc x - \int (-\csc x) dx = -x \csc x + \ln|\csc x - \cot x| + C$$

$$\begin{aligned} 9. \int_1^3 r^4 \ln r dr \quad \left[ \begin{array}{l} u = \ln r, \quad du = \frac{dr}{r} \\ dv = r^4 dr, \quad v = \frac{1}{5}r^5 \end{array} \right] &= [\frac{1}{5}r^5 \ln r]_1^3 - \int_1^3 \frac{1}{5}r^4 dr = \frac{243}{5} \ln 3 - 0 - [\frac{1}{25}r^5]_1^3 \\ &= \frac{243}{5} \ln 3 - (\frac{243}{25} - \frac{1}{25}) = \frac{243}{5} \ln 3 - \frac{242}{25} \end{aligned}$$

10.  $\frac{x-1}{x^2-4x-5} = \frac{x-1}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1} \Rightarrow x-1 = A(x+1) + B(x-5)$ . Setting  $x = -1$  gives

$-2 = -6B$ , so  $B = \frac{1}{3}$ . Setting  $x = 5$  gives  $4 = 6A$ , so  $A = \frac{2}{3}$ . Now

$$\begin{aligned}\int_0^4 \frac{x-1}{x^2-4x-5} dx &= \int_0^4 \left( \frac{2/3}{x-5} + \frac{1/3}{x+1} \right) dx = \left[ \frac{2}{3} \ln|x-5| + \frac{1}{3} \ln|x+1| \right]_0^4 \\ &= \frac{2}{3} \ln 1 + \frac{1}{3} \ln 5 - \frac{2}{3} \ln 5 - \frac{1}{3} \ln 1 = -\frac{1}{3} \ln 5\end{aligned}$$

11.  $\int \frac{x-1}{x^2-4x+5} dx = \int \frac{(x-2)+1}{(x-2)^2+1} dx = \int \left( \frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du \quad [u = x-2, du = dx]$

$$= \frac{1}{2} \ln(u^2+1) + \tan^{-1} u + C = \frac{1}{2} \ln(x^2-4x+5) + \tan^{-1}(x-2) + C$$

12.  $\int \frac{x}{x^4+x^2+1} dx = \int \frac{\frac{1}{2} du}{u^2+u+1} \quad \left[ \begin{array}{l} u = x^2, \\ du = 2x dx \end{array} \right] = \frac{1}{2} \int \frac{du}{(u+\frac{1}{2})^2 + \frac{3}{4}}$

$$= \frac{1}{2} \int \frac{\frac{\sqrt{3}}{2} dv}{\frac{3}{4}(v^2+1)} \quad \left[ \begin{array}{l} u + \frac{1}{2} = \frac{\sqrt{3}}{2} v, \\ du = \frac{\sqrt{3}}{2} dv \end{array} \right] = \frac{\sqrt{3}}{4} \cdot \frac{4}{3} \int \frac{dv}{v^2+1}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} v + C = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2}{\sqrt{3}} (x^2 + \frac{1}{2}) \right) + C$$

13.  $\int \sin^3 \theta \cos^5 \theta d\theta = \int \cos^5 \theta \sin^2 \theta \sin \theta d\theta = - \int \cos^5 \theta (1 - \cos^2 \theta)(-\sin \theta) d\theta$

$$= - \int u^5 (1 - u^2) du \quad \left[ \begin{array}{l} u = \cos \theta, \\ du = -\sin \theta d\theta \end{array} \right]$$

$$= \int (u^7 - u^5) du = \frac{1}{8} u^8 - \frac{1}{6} u^6 + C = \frac{1}{8} \cos^8 \theta - \frac{1}{6} \cos^6 \theta + C$$

Another solution:

$$\begin{aligned}\int \sin^3 \theta \cos^5 \theta d\theta &= \int \sin^3 \theta (\cos^2 \theta)^2 \cos \theta d\theta = \int \sin^3 \theta (1 - \sin^2 \theta)^2 \cos \theta d\theta \\ &= \int u^3 (1 - u^2)^2 du \quad \left[ \begin{array}{l} u = \sin \theta, \\ du = \cos \theta d\theta \end{array} \right] = \int u^3 (1 - 2u^2 + u^4) du \\ &= \int (u^3 - 2u^5 + u^7) du = \frac{1}{4} u^4 - \frac{1}{3} u^6 + \frac{1}{8} u^8 + C = \frac{1}{4} \sin^4 \theta - \frac{1}{3} \sin^6 \theta + \frac{1}{8} \sin^8 \theta + C\end{aligned}$$

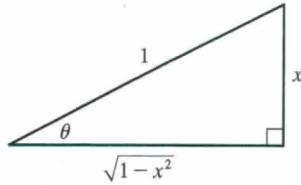
14. Let  $u = 1 + x^2$ , so that  $du = 2x dx$ . Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}} dx &= \int \frac{x^2}{\sqrt{1+x^2}} (x dx) = \int \frac{u-1}{u^{1/2}} (\frac{1}{2} du) = \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du = \frac{1}{2} \left( \frac{2}{3} u^{3/2} - 2u^{1/2} \right) + C \\ &= \frac{1}{3} (1+x^2)^{3/2} - (1+x^2)^{1/2} + C \quad [\text{or } \frac{1}{3} (x^2-2) \sqrt{1+x^2} + C]\end{aligned}$$

15. Let  $x = \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = \cos \theta d\theta$  and  $(1-x^2)^{1/2} = \cos \theta$ ,

so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos \theta d\theta}{(\cos \theta)^3} = \int \sec^2 \theta d\theta = \tan \theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$



$$16. \int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\pi/4} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \quad \begin{bmatrix} u = \sin \theta, \\ du = \cos \theta d\theta \end{bmatrix}$$

$$= \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{1}{2} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/4} = \frac{1}{2} [(\frac{\pi}{4} - \frac{1}{2}) - (0 - 0)] = \frac{\pi}{8} - \frac{1}{4}$$

$$17. \int x \sin^2 x dx \quad \begin{bmatrix} u = x, & dv = \sin^2 x dx, \\ du = dx & v = \int \sin^2 x dx = \int \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}x - \frac{1}{2} \sin x \cos x \end{bmatrix}$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x \sin x \cos x - \int (\frac{1}{2}x - \frac{1}{2} \sin x \cos x) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x \sin x \cos x - \frac{1}{4}x^2 + \frac{1}{4} \sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x \sin x \cos x + \frac{1}{4} \sin^2 x + C$$

Note:  $\int \sin x \cos x dx = \int s ds = \frac{1}{2}s^2 + C$  [where  $s = \sin x$ ,  $ds = \cos x dx$ ].

A slightly different method is to write  $\int x \sin^2 x dx = \int x \cdot \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx$ . If we evaluate the second integral by parts, we arrive at the equivalent answer  $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + C$ .

$$18. \text{ Let } u = e^{2t}, du = 2e^{2t} dt. \text{ Then } \int \frac{e^{2t}}{1+e^{4t}} dt = \int \frac{\frac{1}{2}(2e^{2t}) dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2} du}{1+u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(e^{2t}) + C.$$

$$19. \text{ Let } u = e^x. \text{ Then } \int e^{x+e^x} dx = \int e^{e^x} e^x dx = \int e^u du = e^u + C = e^{e^x} + C.$$

$$20. \text{ Since } e^2 \text{ is a constant, } \int e^2 dx = e^2 x + C.$$

$$21. \text{ Let } t = \sqrt{x}, \text{ so that } t^2 = x \text{ and } 2t dt = dx. \text{ Then } \int \arctan \sqrt{x} dx = \int \arctan t (2t dt) = I. \text{ Now use parts with}$$

$$u = \arctan t, dv = 2t dt \Rightarrow du = \frac{1}{1+t^2} dt, v = t^2. \text{ Thus,}$$

$$I = t^2 \arctan t - \int \frac{t^2}{1+t^2} dt = t^2 \arctan t - \int \left(1 - \frac{1}{1+t^2}\right) dt = t^2 \arctan t - t + \arctan t + C$$

$$= x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C \quad \left[ \text{or } (x+1) \arctan \sqrt{x} - \sqrt{x} + C \right]$$

$$22. \text{ Let } u = 1 + (\ln x)^2, \text{ so that } du = \frac{2 \ln x}{x} dx. \text{ Then}$$

$$\int \frac{\ln x}{x \sqrt{1+(\ln x)^2}} dx = \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u}) + C = \sqrt{1+(\ln x)^2} + C.$$

$$23. \text{ Let } u = 1 + \sqrt{x}. \text{ Then } x = (u-1)^2, dx = 2(u-1) du \Rightarrow$$

$$\int_1^1 (1+\sqrt{x})^8 dx = \int_1^2 u^8 \cdot 2(u-1) du = 2 \int_1^2 (u^9 - u^8) du = \left[\frac{1}{5}u^{10} - 2 \cdot \frac{1}{9}u^9\right]_1^2 = \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}.$$

$$24. \text{ Let } u = \ln(x^2 - 1), dv = dx \Leftrightarrow du = \frac{2x}{x^2 - 1} dx, v = x. \text{ Then}$$

$$\int \ln(x^2 - 1) dx = x \ln(x^2 - 1) - \int \frac{2x^2}{x^2 - 1} dx = x \ln(x^2 - 1) - \int \left[2 + \frac{2}{(x-1)(x+1)}\right] dx$$

$$= x \ln(x^2 - 1) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right] dx = x \ln(x^2 - 1) - 2x - \ln|x-1| + \ln|x+1| + C$$

25.  $\frac{3x^2 - 2}{x^2 - 2x - 8} = 3 + \frac{6x + 22}{(x-4)(x+2)} = 3 + \frac{A}{x-4} + \frac{B}{x+2} \Rightarrow 6x + 22 = A(x+2) + B(x-4)$ . Setting

$x = 4$  gives  $46 = 6A$ , so  $A = \frac{23}{3}$ . Setting  $x = -2$  gives  $10 = -6B$ , so  $B = -\frac{5}{3}$ . Now

$$\int \frac{3x^2 - 2}{x^2 - 2x - 8} dx = \int \left( 3 + \frac{23/3}{x-4} - \frac{5/3}{x+2} \right) dx = 3x + \frac{23}{3} \ln|x-4| - \frac{5}{3} \ln|x+2| + C.$$

26.  $\int \frac{3x^2 - 2}{x^3 - 2x - 8} dx = \int \frac{du}{u} \quad \begin{bmatrix} u = x^3 - 2x - 8, \\ du = (3x^2 - 2) dx \end{bmatrix} = \ln|u| + C = \ln|x^3 - 2x - 8| + C$

27. Let  $u = 1 + e^x$ , so that  $du = e^x dx = (u-1) dx$ . Then  $\int \frac{1}{1+e^x} dx = \int \frac{1}{u} \cdot \frac{du}{u-1} = \int \frac{1}{u(u-1)} du = I$ . Now

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \Rightarrow 1 = A(u-1) + Bu. \text{ Set } u = 1 \text{ to get } 1 = B. \text{ Set } u = 0 \text{ to get } 1 = -A, \text{ so } A = -1.$$

Thus,  $I = \int \left( \frac{-1}{u} + \frac{1}{u-1} \right) du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C$ .

*Another method:* Multiply numerator and denominator by  $e^{-x}$  and let  $u = e^{-x} + 1$ . This gives the answer in the form  $-\ln(e^{-x} + 1) + C$ .

28.  $\int \sin \sqrt{at} dt = \int \sin u \cdot \frac{2}{a} u du \quad [u = \sqrt{at}, u^2 = at, 2u du = a dt] = \frac{2}{a} \int u \sin u du$   
 $= \frac{2}{a} [-u \cos u + \sin u] + C \quad [\text{integration by parts}] = -\frac{2}{a} \sqrt{at} \cos \sqrt{at} + \frac{2}{a} \sin \sqrt{at} + C$   
 $= -2 \sqrt{\frac{t}{a}} \cos \sqrt{at} + \frac{2}{a} \sin \sqrt{at} + C$

29.  $\int_0^5 \frac{3w-1}{w+2} dw = \int_0^5 \left( 3 - \frac{7}{w+2} \right) dw = \left[ 3w - 7 \ln|w+2| \right]_0^5 = 15 - 7 \ln 7 + 7 \ln 2$   
 $= 15 + 7(\ln 2 - \ln 7) = 15 + 7 \ln \frac{2}{7}$

30.  $x^2 - 4x < 0$  on  $[0, 4]$ , so

$$\int_{-2}^2 |x^2 - 4x| dx = \int_{-2}^0 (x^2 - 4x) dx + \int_0^2 (4x - x^2) dx = \left[ \frac{1}{3}x^3 - 2x^2 \right]_{-2}^0 + \left[ 2x^2 - \frac{1}{3}x^3 \right]_0^2$$
 $= 0 - \left( -\frac{8}{3} - 8 \right) + \left( 8 - \frac{8}{3} \right) - 0 = 16$

31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}} dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x dx}{\sqrt{1-x^2}} = \sin^{-1} x - \sqrt{1-x^2} + C.$$

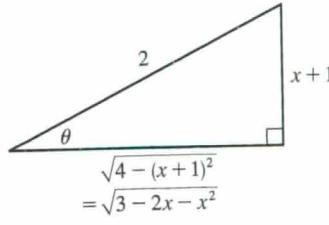
*Another method:* Substitute  $u = \sqrt{(1+x)/(1-x)}$ .

32.  $\int \frac{\sqrt{2x-1}}{2x+3} dx = \int \frac{u \cdot u du}{u^2 + 4} \quad \begin{bmatrix} u = \sqrt{2x-1}, 2x+3 = u^2 + 4, \\ u^2 = 2x-1, u du = dx \end{bmatrix} = \int \left( 1 - \frac{4}{u^2 + 4} \right) du$   
 $= u - 4 \cdot \frac{1}{2} \tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2 \tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$

33.  $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x + 1)^2$ . Let  $x + 1 = 2 \sin \theta$ ,

where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = 2 \cos \theta d\theta$  and

$$\begin{aligned}\int \sqrt{3 - 2x - x^2} dx &= \int \sqrt{4 - (x + 1)^2} dx = \int \sqrt{4 - 4 \sin^2 \theta} 2 \cos \theta d\theta \\ &= 4 \int \cos^2 \theta d\theta = 2 \int (1 + \cos 2\theta) d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2 \sin \theta \cos \theta + C \\ &= 2 \sin^{-1} \left( \frac{x+1}{2} \right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C \\ &= 2 \sin^{-1} \left( \frac{x+1}{2} \right) + \frac{x+1}{2} \sqrt{3-2x-x^2} + C\end{aligned}$$



$$\begin{aligned}34. \int_{\pi/4}^{\pi/2} \frac{1+4 \cot x}{4-\cot x} dx &= \int_{\pi/4}^{\pi/2} \left[ \frac{(1+4 \cos x/\sin x)}{(4-\cos x/\sin x)} \cdot \frac{\sin x}{\sin x} \right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4 \cos x}{4 \sin x - \cos x} dx \\ &= \int_{3/\sqrt{2}}^4 \frac{1}{u} du \quad \left[ \begin{array}{l} u = 4 \sin x - \cos x, \\ du = (4 \cos x + \sin x) dx \end{array} \right] \\ &= \left[ \ln |u| \right]_{3/\sqrt{2}}^4 = \ln 4 - \ln \frac{3}{\sqrt{2}} = \ln \frac{4}{3\sqrt{2}} = \ln \left( \frac{4}{3} \sqrt{2} \right)\end{aligned}$$

35. Because  $f(x) = x^8 \sin x$  is the product of an even function and an odd function, it is odd.

Therefore,  $\int_{-1}^1 x^8 \sin x dx = 0$  [by (5.5.7)(b)].

36.  $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$  by Formula 7.2.2(a), so

$$\int \sin 4x \cos 3x dx = \frac{1}{2} \int (\sin x + \sin 7x) dx = \frac{1}{2} \left[ -\cos x - \frac{1}{7} \cos 7x \right] + C = -\frac{1}{2} \cos x - \frac{1}{14} \cos 7x + C.$$

$$37. \int_0^{\pi/4} \cos^2 \theta \tan^2 \theta d\theta = \int_0^{\pi/4} \sin^2 \theta d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta) d\theta = \left[ \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/4} = \left( \frac{\pi}{8} - \frac{1}{4} \right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

$$\begin{aligned}38. \int_0^{\pi/4} \tan^5 \theta \sec^3 \theta d\theta &= \int_0^{\pi/4} (\tan^2 \theta)^2 \sec^2 \theta \cdot \sec \theta \tan \theta d\theta = \int_1^{\sqrt{2}} (u^2 - 1)^2 u^2 du \quad \left[ \begin{array}{l} u = \sec \theta, \\ du = \sec \theta \tan \theta d\theta \end{array} \right] \\ &= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2) du = \left[ \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 \right]_1^{\sqrt{2}} \\ &= \left( \frac{8}{7}\sqrt{2} - \frac{8}{5}\sqrt{2} + \frac{2}{3}\sqrt{2} \right) - \left( \frac{1}{7} - \frac{2}{5} + \frac{1}{3} \right) = \frac{22}{105}\sqrt{2} - \frac{8}{105} = \frac{2}{105}(11\sqrt{2} - 4)\end{aligned}$$

39. Let  $u = \sec \theta$ , so that  $du = \sec \theta \tan \theta d\theta$ . Then  $\int \frac{\sec \theta \tan \theta}{\sec^2 \theta - \sec \theta} d\theta = \int \frac{1}{u^2 - u} du = \int \frac{1}{u(u-1)} du = I$ . Now

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \Rightarrow 1 = A(u-1) + Bu. \text{ Set } u = 1 \text{ to get } 1 = B. \text{ Set } u = 0 \text{ to get } 1 = -A, \text{ so } A = -1.$$

$$\text{Thus, } I = \int \left( \frac{-1}{u} + \frac{1}{u-1} \right) du = -\ln |u| + \ln |u-1| + C = \ln |\sec \theta - 1| - \ln |\sec \theta| + C \text{ [or } \ln |1 - \cos \theta| + C].$$

40.  $4y^2 - 4y - 3 = (2y - 1)^2 - 2^2$ , so let  $u = 2y - 1 \Rightarrow du = 2 dy$ . Thus,

$$\begin{aligned}\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} &= \int \frac{dy}{\sqrt{(2y-1)^2 - 2^2}} = \frac{1}{2} \int \frac{du}{\sqrt{u^2 - 2^2}} \\ &= \frac{1}{2} \ln |u + \sqrt{u^2 - 2^2}| \quad [\text{by Formula 20 in the table in this section}] \\ &= \frac{1}{2} \ln |2y - 1 + \sqrt{4y^2 - 4y - 3}| + C\end{aligned}$$

41. Let  $u = \theta$ ,  $dv = \tan^2 \theta d\theta = (\sec^2 \theta - 1) d\theta \Rightarrow du = d\theta$  and  $v = \tan \theta - \theta$ . So

$$\begin{aligned}\int \theta \tan^2 \theta d\theta &= \theta(\tan \theta - \theta) - \int (\tan \theta - \theta) d\theta = \theta \tan \theta - \theta^2 - \ln |\sec \theta| + \frac{1}{2} \theta^2 + C \\ &= \theta \tan \theta - \frac{1}{2} \theta^2 - \ln |\sec \theta| + C\end{aligned}$$

42. Let  $u = \tan^{-1} x$ ,  $dv = \frac{1}{x^2} dx \Rightarrow du = \frac{1}{1+x^2} dx$ ,  $v = -\frac{1}{x}$ . Then

$$\begin{aligned}I &= \int \frac{\tan^{-1} x}{x^2} dx = -\frac{1}{x} \tan^{-1} x - \int \left( -\frac{1}{x(1+x^2)} \right) dx = -\frac{1}{x} \tan^{-1} x + \int \left( \frac{A}{x} + \frac{Bx+C}{1+x^2} \right) dx \\ \frac{1}{x(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} \Rightarrow 1 = A(1+x^2) + (Bx+C)x \Rightarrow 1 = (A+B)x^2 + Cx + A, \text{ so } C=0, A=1,\end{aligned}$$

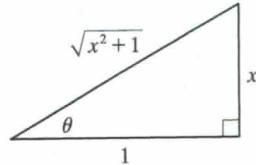
and  $A+B=0 \Rightarrow B=-1$ . Thus,

$$\begin{aligned}I &= -\frac{1}{x} \tan^{-1} x + \int \left( \frac{1}{x} - \frac{x}{1+x^2} \right) dx = -\frac{1}{x} \tan^{-1} x + \ln|x| - \frac{1}{2} \ln|1+x^2| + C \\ &= -\frac{\tan^{-1} x}{x} + \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C\end{aligned}$$

Or: Let  $x = \tan \theta$ , so that  $dx = \sec^2 \theta d\theta$ . Then  $\int \frac{\tan^{-1} x}{x^2} dx = \int \frac{\theta}{\tan^2 \theta} \sec^2 \theta d\theta = \int \theta \csc^2 \theta d\theta = I$ . Now use parts

with  $u = \theta$ ,  $dv = \csc^2 \theta d\theta \Rightarrow du = d\theta$ ,  $v = -\cot \theta$ . Thus,

$$\begin{aligned}I &= -\theta \cot \theta - \int (-\cot \theta) d\theta = -\theta \cot \theta + \ln|\sin \theta| + C \\ &= -\tan^{-1} x \cdot \frac{1}{x} + \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C = -\frac{\tan^{-1} x}{x} + \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C\end{aligned}$$



43. Let  $u = 1 + e^x$ , so that  $du = e^x dx$ . Then  $\int e^x \sqrt{1+e^x} dx = \int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1+e^x)^{3/2} + C$ .

Or: Let  $u = \sqrt{1+e^x}$ , so that  $u^2 = 1 + e^x$  and  $2u du = e^x dx$ . Then

$$\int e^x \sqrt{1+e^x} dx = \int u \cdot 2u du = \int 2u^2 du = \frac{2}{3} u^3 + C = \frac{2}{3} (1+e^x)^{3/2} + C.$$

44. Let  $u = \sqrt{1+e^x}$ . Then  $u^2 = 1 + e^x$ ,  $2u du = e^x dx = (u^2 - 1) dx$ , and  $dx = \frac{2u}{u^2 - 1} du$ , so

$$\begin{aligned}\int \sqrt{1+e^x} dx &= \int u \cdot \frac{2u}{u^2 - 1} du = \int \frac{2u^2}{u^2 - 1} du = \int \left( 2 + \frac{2}{u^2 - 1} \right) du = \int \left( 2 + \frac{1}{u-1} - \frac{1}{u+1} \right) du \\ &= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln(\sqrt{1+e^x} - 1) - \ln(\sqrt{1+e^x} + 1) + C\end{aligned}$$

45. Let  $t = x^3$ . Then  $dt = 3x^2 dx \Rightarrow I = \int x^5 e^{-x^3} dx = \frac{1}{3} \int t e^{-t} dt$ . Now integrate by parts with  $u = t$ ,  $dv = e^{-t} dt$ :

$$I = -\frac{1}{3} t e^{-t} + \frac{1}{3} \int e^{-t} dt = -\frac{1}{3} t e^{-t} - \frac{1}{3} e^{-t} + C = -\frac{1}{3} e^{-x^3} (x^3 + 1) + C.$$

46.  $\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} = \frac{1+2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1+2\sin x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$   
 $= \sec^2 x + 2\sec x \tan x + \tan^2 x = \sec^2 x + 2\sec x \tan x + \sec^2 x - 1 = 2\sec^2 x + 2\sec x \tan x - 1$

Thus,  $\int \frac{1+\sin x}{1-\sin x} dx = \int (2\sec^2 x + 2\sec x \tan x - 1) dx = 2\tan x + 2\sec x - x + C$

47. Let  $u = x - 1$ , so that  $du = dx$ . Then

$$\begin{aligned}\int x^3(x-1)^{-4} dx &= \int (u+1)^3 u^{-4} du = \int (u^3 + 3u^2 + 3u + 1) u^{-4} du = \int (u^{-1} + 3u^{-2} + 3u^{-3} + u^{-4}) du \\ &= \ln|u| - 3u^{-1} - \frac{3}{2}u^{-2} - \frac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \frac{3}{2}(x-1)^{-2} - \frac{1}{3}(x-1)^{-3} + C\end{aligned}$$

48. Let  $u = x^2$ . Then  $du = 2x dx \Rightarrow \int \frac{x dx}{x^4 - a^4} = \int \frac{\frac{1}{2} du}{u^2 - (a^2)^2} = \frac{1}{4a^2} \ln \left| \frac{u-a^2}{u+a^2} \right| + C = \frac{1}{4a^2} \ln \left| \frac{x^2 - a^2}{x^2 + a^2} \right| + C.$

49. Let  $u = \sqrt{4x+1} \Rightarrow u^2 = 4x+1 \Rightarrow 2u du = 4 dx \Rightarrow dx = \frac{1}{2}u du$ . So

$$\begin{aligned}\int \frac{1}{x\sqrt{4x+1}} dx &= \int \frac{\frac{1}{2}u du}{\frac{1}{4}(u^2-1)u} = 2 \int \frac{du}{u^2-1} = 2\left(\frac{1}{2}\right) \ln \left| \frac{u-1}{u+1} \right| + C \quad [\text{by Formula 19}] \\ &= \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right| + C\end{aligned}$$

50. As in Exercise 49, let  $u = \sqrt{4x+1}$ . Then  $\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8 \int \frac{du}{(u^2-1)^2}.$

Now  $\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \Rightarrow$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2. \quad u=1 \Rightarrow D = \frac{1}{4}, u=-1 \Rightarrow B = \frac{1}{4}.$

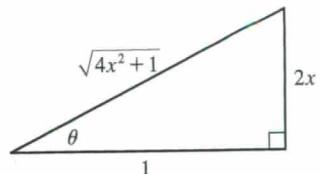
Equating coefficients of  $u^3$  gives  $A+C=0$ , and equating coefficients of 1 gives  $1=A+B-C+D \Rightarrow$

$1 = A + \frac{1}{4} - C + \frac{1}{4} \Rightarrow \frac{1}{2} = A - C. \text{ So } A = \frac{1}{4} \text{ and } C = -\frac{1}{4}. \text{ Therefore,}$

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x+1}} &= 8 \int \left[ \frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2} \right] du \\ &= \int \left[ \frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2} \right] du \\ &= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\ &= 2\ln(\sqrt{4x+1}+1) - \frac{2}{\sqrt{4x+1}+1} - 2\ln|\sqrt{4x+1}-1| - \frac{2}{\sqrt{4x+1}-1} + C\end{aligned}$$

51. Let  $2x = \tan \theta \Rightarrow x = \frac{1}{2} \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{4x^2+1} = \sec \theta$ , so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2} \sec^2 \theta d\theta}{\frac{1}{2} \tan \theta \sec \theta} = \int \frac{\sec \theta}{\tan \theta} d\theta = \int \csc \theta d\theta \\ &= -\ln|\csc \theta + \cot \theta| + C \quad [\text{or } \ln|\csc \theta - \cot \theta| + C] \\ &= -\ln \left| \frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x} \right| + C \quad \left[ \text{or } \ln \left| \frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x} \right| + C \right]\end{aligned}$$



52. Let  $u = x^2$ . Then  $du = 2x \, dx \Rightarrow$

$$\begin{aligned} \int \frac{dx}{x(x^4 + 1)} &= \int \frac{x \, dx}{x^2(x^4 + 1)} = \frac{1}{2} \int \frac{du}{u(u^2 + 1)} = \frac{1}{2} \int \left[ \frac{1}{u} - \frac{u}{u^2 + 1} \right] du = \frac{1}{2} \ln|u| - \frac{1}{4} \ln(u^2 + 1) + C \\ &= \frac{1}{2} \ln(x^2) - \frac{1}{4} \ln(x^4 + 1) + C = \frac{1}{4} [\ln(x^4) - \ln(x^4 + 1)] + C = \frac{1}{4} \ln\left(\frac{x^4}{x^4 + 1}\right) + C \end{aligned}$$

Or: Write  $I = \int \frac{x^3 \, dx}{x^4(x^4 + 1)}$  and let  $u = x^4$ .

53.  $\int x^2 \sinh(mx) \, dx = \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m} \int x \cosh(mx) \, dx \quad \begin{bmatrix} u = x^2, & dv = \sinh(mx) \, dx, \\ du = 2x \, dx, & v = \frac{1}{m} \cosh(mx) \end{bmatrix}$

$$\begin{aligned} &= \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m} \left( \frac{1}{m} x \sinh(mx) - \frac{1}{m} \int \sinh(mx) \, dx \right) \quad \begin{bmatrix} U = x, & dV = \cosh(mx) \, dx, \\ dU = dx, & V = \frac{1}{m} \sinh(mx) \end{bmatrix} \\ &= \frac{1}{m} x^2 \cosh(mx) - \frac{2}{m^2} x \sinh(mx) + \frac{2}{m^3} \cosh(mx) + C \end{aligned}$$

54.  $\int (x + \sin x)^2 \, dx = \int (x^2 + 2x \sin x + \sin^2 x) \, dx = \frac{1}{3} x^3 + 2(\sin x - x \cos x) + \frac{1}{2}(x - \sin x \cos x) + C$   
 $= \frac{1}{3} x^3 + \frac{1}{2} x + 2 \sin x - \frac{1}{2} \sin x \cos x - 2x \cos x + C$

55. Let  $u = \sqrt{x}$ , so that  $x = u^2$  and  $dx = 2u \, du$ . Then  $\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u \, du}{u^2 + u^2 \cdot u} = \int \frac{2}{u(1+u)} \, du = I$ .

Now  $\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \Rightarrow 2 = A(1+u) + Bu$ . Set  $u = -1$  to get  $2 = -B$ , so  $B = -2$ . Set  $u = 0$  to get  $2 = A$ .

Thus,  $I = \int \left( \frac{2}{u} - \frac{2}{1+u} \right) du = 2 \ln|u| - 2 \ln|1+u| + C = 2 \ln \sqrt{x} - 2 \ln(1+\sqrt{x}) + C$ .

56. Let  $u = \sqrt{x}$ , so that  $x = u^2$  and  $dx = 2u \, du$ . Then

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{2u \, du}{u + u^2 \cdot u} = \int \frac{2}{1+u^2} \, du = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x} + C.$$

57. Let  $u = \sqrt[3]{x+c}$ . Then  $x = u^3 - c \Rightarrow$

$$\int x \sqrt[3]{x+c} \, dx = \int (u^3 - c)u \cdot 3u^2 \, du = 3 \int (u^6 - cu^3) \, du = \frac{3}{7} u^7 - \frac{3}{4} cu^4 + C = \frac{3}{7} (x+c)^{7/3} - \frac{3}{4} c(x+c)^{4/3} + C$$

58. Let  $t = \sqrt{x^2 - 1}$ . Then  $dt = (x/\sqrt{x^2 - 1}) \, dx$ ,  $x^2 - 1 = t^2$ ,  $x = \sqrt{t^2 + 1}$ , so

$I = \int \frac{x \ln x}{\sqrt{x^2 - 1}} \, dx = \int \ln \sqrt{t^2 + 1} \, dt = \frac{1}{2} \int \ln(t^2 + 1) \, dt$ . Now use parts with  $u = \ln(t^2 + 1)$ ,  $dv = dt$ :

$$\begin{aligned} I &= \frac{1}{2} t \ln(t^2 + 1) - \int \frac{t^2}{t^2 + 1} \, dt = \frac{1}{2} t \ln(t^2 + 1) - \int \left[ 1 - \frac{1}{t^2 + 1} \right] dt \\ &= \frac{1}{2} t \ln(t^2 + 1) - t + \tan^{-1} t + C = \sqrt{x^2 - 1} \ln x - \sqrt{x^2 - 1} + \tan^{-1} \sqrt{x^2 - 1} + C \end{aligned}$$

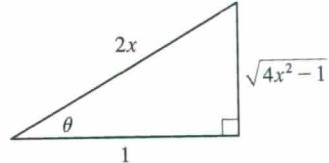
Another method: First integrate by parts with  $u = \ln x$ ,  $dv = (x/\sqrt{x^2 - 1}) \, dx$  and then use substitution  
 $(x = \sec \theta \text{ or } u = \sqrt{x^2 - 1})$ .

59. Let  $u = \sin x$ , so that  $du = \cos x dx$ . Then

$$\begin{aligned}\int \cos x \cos^3(\sin x) dx &= \int \cos^3 u du = \int \cos^2 u \cos u du = \int (1 - \sin^2 u) \cos u du \\ &= \int (\cos u - \sin^2 u \cos u) du = \sin u - \frac{1}{3} \sin^3 u + C = \sin(\sin x) - \frac{1}{3} \sin^3(\sin x) + C\end{aligned}$$

60. Let  $2x = \sec \theta$ , so that  $2dx = \sec \theta \tan \theta d\theta$ . Then

$$\begin{aligned}\int \frac{dx}{x^2 \sqrt{4x^2 - 1}} &= \int \frac{\frac{1}{2} \sec \theta \tan \theta d\theta}{\frac{1}{4} \sec^2 \theta \sqrt{\sec^2 \theta - 1}} = \int \frac{2 \tan \theta d\theta}{\sec \theta \tan \theta} \\ &= 2 \int \cos \theta d\theta = 2 \sin \theta + C \\ &= 2 \cdot \frac{\sqrt{4x^2 - 1}}{2x} + C = \frac{\sqrt{4x^2 - 1}}{x} + C\end{aligned}$$



61. Let  $y = \sqrt{x}$  so that  $dy = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} dy = 2y dy$ . Then

$$\begin{aligned}\int \sqrt{x} e^{\sqrt{x}} dx &= \int ye^y (2y dy) = \int 2y^2 e^y dy \quad \left[ \begin{array}{l} u = 2y^2, \quad dv = e^y dy, \\ du = 4y dy \quad v = e^y \end{array} \right] \\ &= 2y^2 e^y - \int 4ye^y dy \quad \left[ \begin{array}{l} U = 4y, \quad dV = e^y dy, \\ dU = 4 dy \quad V = e^y \end{array} \right] \\ &= 2y^2 e^y - (4ye^y - \int 4e^y dy) = 2y^2 e^y - 4ye^y + 4e^y + C \\ &= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)e^{\sqrt{x}} + C\end{aligned}$$

62. Let  $u = \sqrt[3]{x}$ . Then  $x = u^3$ ,  $dx = 3u^2 du \Rightarrow$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2 du}{u^3 + u} = \frac{3}{2} \int \frac{2u du}{u^2 + 1} = \frac{3}{2} \ln(u^2 + 1) + C = \frac{3}{2} \ln(x^{2/3} + 1) + C.$$

63. Let  $u = \cos^2 x$ , so that  $du = 2 \cos x (-\sin x) dx$ . Then

$$\int \frac{\sin 2x}{1 + \cos^4 x} dx = \int \frac{2 \sin x \cos x}{1 + (\cos^2 x)^2} dx = \int \frac{1}{1 + u^2} (-du) = -\tan^{-1} u + C = -\tan^{-1}(\cos^2 x) + C.$$

64. Let  $u = \tan x$ . Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x) dx}{\sin x \cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x} \sec^2 x dx = \int_1^{\sqrt{3}} \frac{\ln u}{u} du = [\frac{1}{2}(\ln u)^2]_1^{\sqrt{3}} = \frac{1}{2} (\ln \sqrt{3})^2 = \frac{1}{8} (\ln 3)^2.$$

$$\begin{aligned}65. \int \frac{dx}{\sqrt{x+1} + \sqrt{x}} &= \int \left( \frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \right) dx = \int (\sqrt{x+1} - \sqrt{x}) dx \\ &= \frac{2}{3} [(x+1)^{3/2} - x^{3/2}] + C\end{aligned}$$

$$66. \int \frac{u^3 + 1}{u^3 - u^2} du = \int \left[ 1 + \frac{u^2 + 1}{(u-1)u^2} \right] du = u + \int \left[ \frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2} \right] du = u + 2 \ln |u-1| - \ln |u| + \frac{1}{u} + C. \text{ Thus,}$$

$$\begin{aligned}\int_2^3 \frac{u^3 + 1}{u^3 - u^2} du &= \left[ u + 2 \ln(u-1) - \ln u + \frac{1}{u} \right]_2^3 = (3 + 2 \ln 2 - \ln 3 + \frac{1}{3}) - (2 + 2 \ln 1 - \ln 2 + \frac{1}{2}) \\ &= 1 + 3 \ln 2 - \ln 3 - \frac{1}{6} = \frac{5}{6} + \ln \frac{8}{3}\end{aligned}$$

67. Let  $x = \tan \theta$ , so that  $dx = \sec^2 \theta d\theta$ ,  $x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ , and  $x = 1 \Rightarrow \theta = \frac{\pi}{4}$ . Then

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx &= \int_{\pi/4}^{\pi/3} \frac{\sec \theta}{\tan^2 \theta} \sec^2 \theta d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec \theta (\tan^2 \theta + 1)}{\tan^2 \theta} d\theta = \int_{\pi/4}^{\pi/3} \left( \frac{\sec \theta \tan^2 \theta}{\tan^2 \theta} + \frac{\sec \theta}{\tan^2 \theta} \right) d\theta \\ &= \int_{\pi/4}^{\pi/3} (\sec \theta + \csc \theta \cot \theta) d\theta = \left[ \ln |\sec \theta + \tan \theta| - \csc \theta \right]_{\pi/4}^{\pi/3} \\ &= \left( \ln |2 + \sqrt{3}| - \frac{2}{\sqrt{3}} \right) - \left( \ln |\sqrt{2} + 1| - \sqrt{2} \right) = \sqrt{2} - \frac{2}{\sqrt{3}} + \ln(2 + \sqrt{3}) - \ln(1 + \sqrt{2}) \end{aligned}$$

68. Let  $u = e^x$ . Then  $x = \ln u$ ,  $dx = du/u \Rightarrow$

$$\begin{aligned} \int \frac{dx}{1+2e^x-e^{-x}} &= \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[ \frac{2/3}{2u-1} - \frac{1/3}{u+1} \right] du \\ &= \frac{1}{3} \ln|2u-1| - \frac{1}{3} \ln|u+1| + C = \frac{1}{3} \ln|(2e^x-1)/(e^x+1)| + C \end{aligned}$$

69. Let  $u = e^x$ . Then  $x = \ln u$ ,  $dx = du/u \Rightarrow$

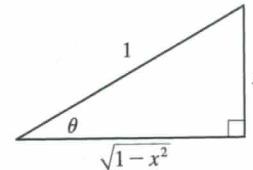
$$\int \frac{e^{2x}}{1+e^x} dx = \int \frac{u^2}{1+u} \frac{du}{u} = \int \frac{u}{1+u} du = \int \left( 1 - \frac{1}{1+u} \right) du = u - \ln|1+u| + C = e^x - \ln(1+e^x) + C.$$

70. Use parts with  $u = \ln(x+1)$ ,  $dv = dx/x^2$ :

$$\begin{aligned} \int \frac{\ln(x+1)}{x^2} dx &= -\frac{1}{x} \ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x} \ln(x+1) + \int \left[ \frac{1}{x} - \frac{1}{x+1} \right] dx \\ &= -\frac{1}{x} \ln(x+1) + \ln|x| - \ln(x+1) + C = -\left( 1 + \frac{1}{x} \right) \ln(x+1) + \ln|x| + C \end{aligned}$$

71. Let  $\theta = \arcsin x$ , so that  $d\theta = \frac{1}{\sqrt{1-x^2}} dx$  and  $x = \sin \theta$ . Then

$$\begin{aligned} \int \frac{x+\arcsin x}{\sqrt{1-x^2}} dx &= \int (\sin \theta + \theta) d\theta = -\cos \theta + \frac{1}{2}\theta^2 + C \\ &= -\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C \end{aligned}$$



$$72. \int \frac{4^x + 10^x}{2^x} dx = \int \left( \frac{4^x}{2^x} + \frac{10^x}{2^x} \right) dx = \int (2^x + 5^x) dx = \frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + C$$

$$73. \frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \Rightarrow 1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C).$$

So  $0 = A + B = C - 2B$ ,  $1 = 4A - 2C$ . Setting  $x = 2$  gives  $A = \frac{1}{8} \Rightarrow B = -\frac{1}{8}$  and  $C = -\frac{1}{4}$ . So

$$\begin{aligned} \int \frac{1}{(x-2)(x^2+4)} dx &= \int \left( \frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4} \right) dx = \frac{1}{8} \int \frac{dx}{x-2} - \frac{1}{16} \int \frac{2x dx}{x^2+4} - \frac{1}{4} \int \frac{dx}{x^2+4} \\ &= \frac{1}{8} \ln|x-2| - \frac{1}{16} \ln(x^2+4) - \frac{1}{8} \tan^{-1}(x/2) + C \end{aligned}$$

74. Let  $u = 2 + \sqrt{x}$ , so that  $du = \frac{1}{2\sqrt{x}} dx$ . Then

$$\int \frac{dx}{\sqrt{x}(2+\sqrt{x})^4} = \int \frac{2du}{u^4} = 2 \int u^{-4} du = -\frac{2}{3}u^{-3} + C = -\frac{2}{3(2+\sqrt{x})^3} + C.$$

75. Let  $y = \sqrt{1+e^x}$ , so that  $y^2 = 1+e^x$ ,  $2y dy = e^x dx$ ,  $e^x = y^2 - 1$ , and  $x = \ln(y^2 - 1)$ . Then

$$\begin{aligned} \int \frac{xe^x}{\sqrt{1+e^x}} dx &= \int \frac{\ln(y^2-1)}{y} (2y dy) = 2 \int [\ln(y+1) + \ln(y-1)] dy \\ &= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C \quad [\text{by Example 7.1.2}] \\ &= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C \\ &= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C \\ &= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x} \ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C \\ &= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C \end{aligned}$$

76.  $\int (x^2 - bx) \sin 2x dx = -\frac{1}{2}(x^2 - bx) \cos 2x + \frac{1}{2} \int (2x - b) \cos 2x dx$   
 $[u = x^2 - bx, dv = \sin 2x dx, du = (2x - b) dx, v = -\frac{1}{2} \cos 2x]$   
 $= -\frac{1}{2}(x^2 - bx) \cos 2x + \frac{1}{2} \left[ \frac{1}{2}(2x - b) \sin 2x - \int \sin 2x dx \right]$   
 $[U = 2x - b, dV = \cos 2x dx, dU = 2 dx, V = \frac{1}{2} \sin 2x]$   
 $= -\frac{1}{2}(x^2 - bx) \cos 2x + \frac{1}{4}(2x - b) \sin 2x + \frac{1}{4} \cos 2x + C$

77. Let  $u = x^{3/2}$  so that  $u^2 = x^3$  and  $du = \frac{3}{2}x^{1/2} dx \Rightarrow \sqrt{x} dx = \frac{2}{3} du$ . Then

$$\int \frac{\sqrt{x}}{1+x^3} dx = \int \frac{\frac{2}{3}}{1+u^2} du = \frac{2}{3} \tan^{-1} u + C = \frac{2}{3} \tan^{-1}(x^{3/2}) + C.$$

78.  $\int \frac{\sec x \cos 2x}{\sin x + \sec x} dx = \int \frac{\sec x \cos 2x}{\sin x + \sec x} \cdot \frac{2 \cos x}{2 \cos x} dx = \int \frac{2 \cos 2x}{2 \sin x \cos x + 2} dx$   
 $= \int \frac{2 \cos 2x}{\sin 2x + 2} dx = \int \frac{1}{u} du \quad \begin{bmatrix} u = \sin 2x + 2, \\ du = 2 \cos 2x dx \end{bmatrix}$   
 $= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C$

79. Let  $u = x$ ,  $dv = \sin^2 x \cos x dx \Rightarrow du = dx$ ,  $v = \frac{1}{3} \sin^3 x$ . Then

$$\begin{aligned} \int x \sin^2 x \cos x dx &= \frac{1}{3}x \sin^3 x - \int \frac{1}{3} \sin^3 x dx = \frac{1}{3}x \sin^3 x - \frac{1}{3} \int (1 - \cos^2 x) \sin x dx \\ &= \frac{1}{3}x \sin^3 x + \frac{1}{3} \int (1 - y^2) dy \quad \begin{bmatrix} u = \cos x, \\ du = -\sin x dx \end{bmatrix} \\ &= \frac{1}{3}x \sin^3 x + \frac{1}{3}y - \frac{1}{9}y^3 + C = \frac{1}{3}x \sin^3 x + \frac{1}{3} \cos x - \frac{1}{9} \cos^3 x + C \end{aligned}$$

$$\begin{aligned}
 80. \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \int \frac{\sin x \cos x}{(\sin^2 x)^2 + (\cos^2 x)^2} dx = \int \frac{\sin x \cos x}{(\sin^2 x)^2 + (1 - \sin^2 x)^2} dx \\
 &= \int \frac{1}{u^2 + (1-u)^2} \left( \frac{1}{2} du \right) \quad \left[ \begin{array}{l} u = \sin^2 x, \\ du = 2 \sin x \cos x dx \end{array} \right] \\
 &= \int \frac{1}{4u^2 - 4u + 2} du = \int \frac{1}{(4u^2 - 4u + 1) + 1} du \\
 &= \int \frac{1}{(2u-1)^2 + 1} du = \frac{1}{2} \int \frac{1}{y^2 + 1} dy \quad \left[ \begin{array}{l} y = 2u-1, \\ dy = 2 du \end{array} \right] \\
 &= \frac{1}{2} \tan^{-1} y + C = \frac{1}{2} \tan^{-1}(2u-1) + C = \frac{1}{2} \tan^{-1}(2 \sin^2 x - 1) + C
 \end{aligned}$$

Another solution:

$$\begin{aligned}
 \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx &= \int \frac{(\sin x \cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x} dx = \int \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx \\
 &= \int \frac{1}{u^2 + 1} \left( \frac{1}{2} du \right) \quad \left[ \begin{array}{l} u = \tan^2 x, \\ du = 2 \tan x \sec^2 x dx \end{array} \right] \\
 &= \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1}(\tan^2 x) + C
 \end{aligned}$$

81. The function  $y = 2xe^{x^2}$  does have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\begin{aligned}
 \int (2x^2 + 1)e^{x^2} dx &= \int 2x^2 e^{x^2} dx + \int e^{x^2} dx = \int x(2xe^{x^2}) dx + \int e^{x^2} dx \\
 &= xe^{x^2} - \int e^{x^2} dx + \int e^{x^2} dx \quad \left[ \begin{array}{l} u = x, \quad dv = 2xe^{x^2} dx, \\ du = dx \quad v = e^{x^2} \end{array} \right] = xe^{x^2} + C
 \end{aligned}$$

## 7.6 Integration Using Tables and Computer Algebra Systems

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Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution  $u = \sqrt{2}x$  to obtain the radical  $\sqrt{7-u^2}$  and then use Formula 33 with  $a = \sqrt{7}$ .

Alternatively, we will factor  $\sqrt{2}$  out of the radical and use  $a = \sqrt{\frac{7}{2}}$ .

$$\begin{aligned}
 \int \frac{\sqrt{7-2x^2}}{x^2} dx &= \sqrt{2} \int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2} dx \stackrel{33}{=} \sqrt{2} \left[ -\frac{1}{x} \sqrt{\frac{7}{2}-x^2} - \sin^{-1} \frac{x}{\sqrt{\frac{7}{2}}} \right] + C \\
 &= -\frac{1}{x} \sqrt{7-2x^2} - \sqrt{2} \sin^{-1} \left( \sqrt{\frac{2}{7}} x \right) + C
 \end{aligned}$$

$$\begin{aligned}
 2. \int \frac{3x}{\sqrt{3-2x}} dx &= 3 \int \frac{x}{\sqrt{3+(-2)x}} dx \stackrel{55}{=} 3 \left[ \frac{2}{3(-2)^2} (-2x - 2 \cdot 3) \sqrt{3+(-2)x} \right] + C \\
 &= \frac{1}{2} (-2x - 6) \sqrt{3-2x} + C = -(x+3) \sqrt{3-2x} + C
 \end{aligned}$$

3. Let  $u = \pi x \Rightarrow du = \pi dx$ , so

$$\begin{aligned}
 \int \sec^3(\pi x) dx &= \frac{1}{\pi} \int \sec^3 u du \stackrel{71}{=} \frac{1}{\pi} \left( \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| \right) + C \\
 &= \frac{1}{2\pi} \sec \pi x \tan \pi x + \frac{1}{2\pi} \ln |\sec \pi x + \tan \pi x| + C
 \end{aligned}$$