7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}\,dx = \int \left(1+\frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2)+B(x+5)\,(*)$, or equivalently, $x-9=(A+B)x-2A+5B$. Equating coefficients of x on each side of the equation gives us $1=A+B$ **(1)** and equating constants gives us $-9=-2A+5B$ **(2)**. Adding two times **(1)** to **(2)** gives us $-7=7B \quad\Leftrightarrow\quad B=-1$ and hence, $A=2$. [Alternatively, to find the coefficients A and B, we may use substitution as follows: substitute 2 for x in $(*)$ to get $-7=7B \quad\Leftrightarrow\quad B=-1$, then substitute -5 for x in $(*)$ to get $-14=-7A \quad\Leftrightarrow\quad A=2$.] Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int\left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx = 2\ln|x+5| - \ln|x-2| + C.$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1}+\dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1=A(x-1)+B(x+1)$. Substituting 1 for x gives $1=2B \quad\Leftrightarrow\quad B=\frac{1}{2}$. Substituting -1 for x gives $1=-2A \quad\Leftrightarrow\quad A=-\frac{1}{2}$. Thus,

$$\begin{aligned}\int_2^3 \frac{1}{x^2-1}\,dx &= \int_2^3\left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1|+\tfrac{1}{2}\ln|x-1|\right]_2^3 \\ &= \left(-\tfrac{1}{2}\ln 4+\tfrac{1}{2}\ln 2\right)-\left(-\tfrac{1}{2}\ln 3+\tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2+\ln 3-\ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]\end{aligned}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12 = A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.

Now

$$\begin{aligned}\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy &= \int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy = \left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2 \\ &= 2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2 \\ &= 2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2-\ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}\end{aligned}$$

25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get $10 = A(x^2+9)+(Bx+C)(x-1)\ (\star)$. Substituting 1 for x gives $10=10A \quad\Leftrightarrow\quad A=1$. Substituting 0 for x gives $10=9A-C \quad\Rightarrow\quad C=9(1)-10=-1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0=A+B \quad\Rightarrow\quad B=-1$. Thus,

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx \\ &= \ln|x-1|-\tfrac{1}{2}\ln(x^2+9)-\tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)+C\end{aligned}$$

In the second term we used the substitution $u=x^2+9$ and in the last term we used Formula 10.

29. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$

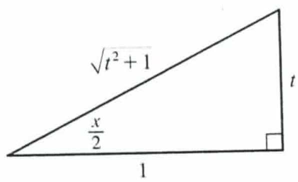
$$= \frac{1}{2}\ln\left|x^2+2x+5\right| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{l}\text{where } x+1=2u,\\ \text{and } dx = 2\,du\end{array}\right]$$

$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

39. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u\,du \;\Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u\,du}{(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

57. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1}t$. The figure gives



$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

(b) $\cos x = \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$$= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

(c) $\frac{x}{2} = \arctan t \;\Rightarrow\; x = 2\arctan t \;\Rightarrow\; dx = \frac{2}{1+t^2}\,dt$

59. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 57, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t)-4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5}\,\frac{1}{2t-1} - \frac{1}{5}\,\frac{1}{t+2}\right] dt \qquad \text{[using partial fractions]}$$

$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$