

68.  $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] dx.$

If  $m \neq n$ , this is equal to  $\frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0.$

If  $m = n$ , we get  $\int_{-\pi}^{\pi} \frac{1}{2} [1 - \cos(m+n)x] dx = \left[ \frac{1}{2}x \right]_{-\pi}^{\pi} - \left[ \frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi - 0 = \pi.$

69.  $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] dx.$

If  $m \neq n$ , this is equal to  $\frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0.$

If  $m = n$ , we get  $\int_{-\pi}^{\pi} \frac{1}{2} [1 + \cos(m+n)x] dx = \left[ \frac{1}{2}x \right]_{-\pi}^{\pi} + \left[ \frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi + 0 = \pi.$

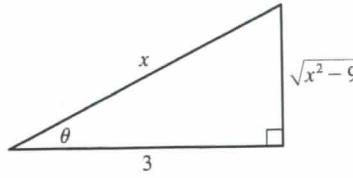
70.  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \left( \sum_{n=1}^m a_n \sin nx \right) \sin mx \right] dx = \sum_{n=1}^m \frac{a_n}{\pi} \int_{-\pi}^{\pi} \sin mx \sin nx dx.$  By Exercise 68, every term is zero except the  $m$ th one, and that term is  $\frac{a_m}{\pi} \cdot \pi = a_m.$

### 7.3 Trigonometric Substitution

1. Let  $x = 3 \sec \theta$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ . Then

$$dx = 3 \sec \theta \tan \theta d\theta \text{ and}$$

$$\begin{aligned} \sqrt{x^2 - 9} &= \sqrt{9 \sec^2 \theta - 9} = \sqrt{9(\sec^2 \theta - 1)} = \sqrt{9 \tan^2 \theta} \\ &= 3 |\tan \theta| = 3 \tan \theta \text{ for the relevant values of } \theta. \end{aligned}$$

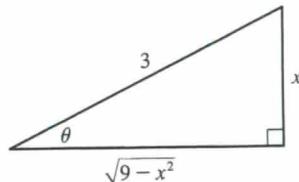


$$\int \frac{1}{x^2 \sqrt{x^2 - 9}} dx = \int \frac{1}{9 \sec^2 \theta \cdot 3 \tan \theta} 3 \sec \theta \tan \theta d\theta = \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + C = \frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C$$

Note that  $-\sec(\theta + \pi) = \sec \theta$ , so the figure is sufficient for the case  $\pi \leq \theta < \frac{3\pi}{2}$ .

2. Let  $x = 3 \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = 3 \cos \theta d\theta$  and

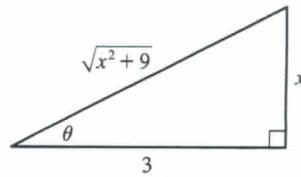
$$\begin{aligned} \sqrt{9 - x^2} &= \sqrt{9 - 9 \sin^2 \theta} = \sqrt{9(1 - \sin^2 \theta)} = \sqrt{9 \cos^2 \theta} \\ &= 3 |\cos \theta| = 3 \cos \theta \text{ for the relevant values of } \theta. \end{aligned}$$



$$\begin{aligned} \int x^3 \sqrt{9 - x^2} dx &= \int 3^3 \sin^3 \theta \cdot 3 \cos \theta \cdot 3 \cos \theta d\theta = 3^5 \int \sin^3 \theta \cos^2 \theta d\theta = 3^5 \int \sin^2 \theta \cos^2 \theta \sin \theta d\theta \\ &= 3^5 \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta = 3^5 \int (1 - u^2) u^2 (-du) \quad [u = \cos \theta, du = -\sin \theta d\theta] \\ &= 3^5 \int (u^4 - u^2) du = 3^5 \left( \frac{1}{5} u^5 - \frac{1}{3} u^3 \right) + C = 3^5 \left( \frac{1}{5} \cos^5 \theta - \frac{1}{3} \cos^3 \theta \right) + C \\ &= 3^5 \left[ \frac{1}{5} \frac{(9 - x^2)^{5/2}}{3^5} - \frac{1}{3} \frac{(9 - x^2)^{3/2}}{3^3} \right] + C \\ &= \frac{1}{5} (9 - x^2)^{5/2} - 3(9 - x^2)^{3/2} + C \quad \text{or} \quad -\frac{1}{5} (x^2 + 6)(9 - x^2)^{3/2} + C \end{aligned}$$

3. Let  $x = 3 \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $dx = 3 \sec^2 \theta d\theta$  and

$$\begin{aligned}\sqrt{x^2 + 9} &= \sqrt{9 \tan^2 \theta + 9} = \sqrt{9(\tan^2 \theta + 1)} = \sqrt{9 \sec^2 \theta} \\ &= 3 |\sec \theta| = 3 \sec \theta \text{ for the relevant values of } \theta.\end{aligned}$$



$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2 + 9}} dx &= \int \frac{3^3 \tan^3 \theta}{3 \sec \theta} 3 \sec^2 \theta d\theta = 3^3 \int \tan^3 \theta \sec \theta d\theta = 3^3 \int \tan^2 \theta \tan \theta \sec \theta d\theta \\ &= 3^3 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta = 3^3 \int (u^2 - 1) du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] \\ &= 3^3 \left( \frac{1}{3} u^3 - u \right) + C = 3^3 \left( \frac{1}{3} \sec^3 \theta - \sec \theta \right) + C = 3^3 \left[ \frac{1}{3} \frac{(x^2 + 9)^{3/2}}{3^3} - \frac{\sqrt{x^2 + 9}}{3} \right] + C \\ &= \frac{1}{3} (x^2 + 9)^{3/2} - 9 \sqrt{x^2 + 9} + C \quad \text{or} \quad \frac{1}{3} (x^2 - 18) \sqrt{x^2 + 9} + C\end{aligned}$$

4. Let  $x = 4 \sin \theta$ , where  $-\pi/2 \leq \theta \leq \pi/2$ . Then  $dx = 4 \cos \theta d\theta$  and

$$\sqrt{16 - x^2} = \sqrt{16 - 16 \sin^2 \theta} = \sqrt{16 \cos^2 \theta} = 4 |\cos \theta| = 4 \cos \theta. \text{ When } x = 0, 4 \sin \theta = 0 \Rightarrow \theta = 0,$$

and when  $x = 2\sqrt{3}$ ,  $4 \sin \theta = 2\sqrt{3} \Rightarrow \sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$ . Thus, substitution gives

$$\begin{aligned}\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16 - x^2}} dx &= \int_0^{\pi/3} \frac{4^3 \sin^3 \theta}{4 \cos \theta} 4 \cos \theta d\theta = 4^3 \int_0^{\pi/3} \sin^3 \theta d\theta = 4^3 \int_0^{\pi/3} (1 - \cos^2 \theta) \sin \theta d\theta \\ &\stackrel{u}{=} -4^3 \int_1^{1/2} (1 - u^2) du = -64 \left[ u - \frac{1}{3} u^3 \right]_1^{1/2} \\ &= -64 \left[ \left( \frac{1}{2} - \frac{1}{24} \right) - \left( 1 - \frac{1}{3} \right) \right] = -64 \left( -\frac{5}{24} \right) = \frac{40}{3}\end{aligned}$$

Or: Let  $u = 16 - x^2$ ,  $x^2 = 16 - u$ ,  $du = -2x dx$ .

5. Let  $t = \sec \theta$ , so  $dt = \sec \theta \tan \theta d\theta$ ,  $t = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ , and  $t = 2 \Rightarrow \theta = \frac{\pi}{3}$ . Then

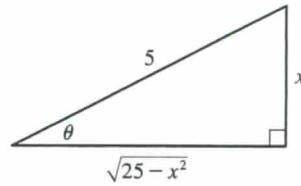
$$\begin{aligned}\int_{\sqrt{2}}^2 \frac{1}{t^3 \sqrt{t^2 - 1}} dt &= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3 \theta \tan \theta} \sec \theta \tan \theta d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2 \theta} d\theta = \int_{\pi/4}^{\pi/3} \cos^2 \theta d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{\pi/4}^{\pi/3} \\ &= \frac{1}{2} \left[ \left( \frac{\pi}{3} + \frac{1}{2} \frac{\sqrt{3}}{2} \right) - \left( \frac{\pi}{4} + \frac{1}{2} \cdot 1 \right) \right] = \frac{1}{2} \left( \frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2} \right) = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}\end{aligned}$$

6. Let  $x = \sec \theta$ , so  $dx = \sec \theta \tan \theta d\theta$ ,  $x = 1 \Rightarrow \theta = 0$ , and  $x = 2 \Rightarrow \theta = \frac{\pi}{3}$ . Then

$$\begin{aligned}\int_1^2 \frac{\sqrt{x^2 - 1}}{x} dx &= \int_0^{\pi/3} \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta = \int_0^{\pi/3} \tan^2 \theta d\theta = \int_0^{\pi/3} (\sec^2 \theta - 1) d\theta \\ &= [\tan \theta - \theta]_0^{\pi/3} = \left( \sqrt{3} - \frac{\pi}{3} \right) - 0 = \sqrt{3} - \frac{\pi}{3}\end{aligned}$$

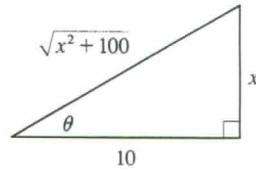
7. Let  $x = 5 \sin \theta$ , so  $dx = 5 \cos \theta d\theta$ . Then

$$\begin{aligned}\int \frac{1}{x^2 \sqrt{25 - x^2}} dx &= \int \frac{1}{5^2 \sin^2 \theta \cdot 5 \cos \theta} 5 \cos \theta d\theta = \frac{1}{25} \int \csc^2 \theta d\theta \\ &= -\frac{1}{25} \cot \theta + C = -\frac{1}{25} \frac{\sqrt{25 - x^2}}{x} + C\end{aligned}$$



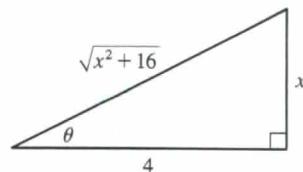
8. Let  $x = 10 \tan \theta$ , so  $dx = 10 \sec^2 \theta d\theta$ . Then

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 100}} dx &= \int \frac{1000 \tan^3 \theta}{10 \sec \theta} 10 \sec^2 \theta d\theta \\ &= 1000 \int \tan^3 \theta \sec \theta d\theta = 1000 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= 1000 \int (u^2 - 1) du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] \\ &= 1000 \left( \frac{1}{3} u^3 - u \right) + C = \frac{1000}{3} u(u^2 - 3) + C = \frac{1000}{3} \sec \theta (\sec^2 \theta - 3) + C \\ &= \frac{1000}{3} \frac{\sqrt{x^2 + 100}}{10} \left( \frac{x^2 + 100}{100} - 3 \right) + C = \frac{100}{3} \sqrt{x^2 + 100} \frac{x^2 - 200}{100} + C \\ &= \frac{1}{3}(x^2 - 200) \sqrt{x^2 + 100} + C \end{aligned}$$



9. Let  $x = 4 \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $dx = 4 \sec^2 \theta d\theta$  and

$$\begin{aligned} \sqrt{x^2 + 16} &= \sqrt{16 \tan^2 \theta + 16} = \sqrt{16(\tan^2 \theta + 1)} \\ &= \sqrt{16 \sec^2 \theta} = 4 |\sec \theta| \\ &= 4 \sec \theta \text{ for the relevant values of } \theta. \end{aligned}$$

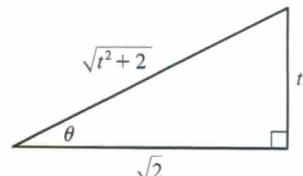


$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 16}} &= \int \frac{4 \sec^2 \theta d\theta}{4 \sec \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 = \ln \left| \frac{\sqrt{x^2 + 16}}{4} + \frac{x}{4} \right| + C_1 \\ &= \ln |\sqrt{x^2 + 16} + x| - \ln |4| + C_1 = \ln(\sqrt{x^2 + 16} + x) + C, \text{ where } C = C_1 - \ln 4. \end{aligned}$$

(Since  $\sqrt{x^2 + 16} + x > 0$ , we don't need the absolute value.)

10. Let  $t = \sqrt{2} \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $dt = \sqrt{2} \sec^2 \theta d\theta$  and

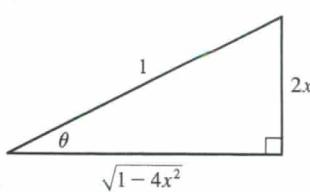
$$\begin{aligned} \sqrt{t^2 + 2} &= \sqrt{2 \tan^2 \theta + 2} = \sqrt{2(\tan^2 \theta + 1)} = \sqrt{2 \sec^2 \theta} \\ &= \sqrt{2} |\sec \theta| = \sqrt{2} \sec \theta \text{ for the relevant values of } \theta. \end{aligned}$$



$$\begin{aligned} \int \frac{t^5}{\sqrt{t^2 + 2}} dt &= \int \frac{4 \sqrt{2} \tan^5 \theta}{\sqrt{2} \sec \theta} \sqrt{2} \sec^2 \theta d\theta = 4 \sqrt{2} \int \tan^5 \theta \sec \theta d\theta \\ &= 4 \sqrt{2} \int (\sec^2 \theta - 1)^2 \sec \theta \tan \theta d\theta = 4 \sqrt{2} \int (u^2 - 1)^2 du \quad [u = \sec \theta, du = \sec \theta \tan \theta d\theta] \\ &= 4 \sqrt{2} \int (u^4 - 2u^2 + 1) du = 4 \sqrt{2} \left( \frac{1}{5} u^5 - \frac{2}{3} u^3 + u \right) + C \\ &= \frac{4 \sqrt{2}}{15} u(3u^4 - 10u^2 + 15) + C = \frac{4 \sqrt{2}}{15} \cdot \frac{\sqrt{t^2 + 2}}{\sqrt{2}} \left[ 3 \cdot \frac{(t^2 + 2)^2}{2^2} - 10 \frac{t^2 + 2}{2} + 15 \right] + C \\ &= \frac{4}{15} \sqrt{t^2 + 2} \cdot \frac{1}{4} [3(t^4 + 4t^2 + 4) - 20(t^2 + 2) + 60] + C = \frac{1}{15} \sqrt{t^2 + 2} (3t^4 - 8t^2 + 32) + C \end{aligned}$$

11. Let  $2x = \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $x = \frac{1}{2} \sin \theta$ ,  $dx = \frac{1}{2} \cos \theta d\theta$ , and  $\sqrt{1 - 4x^2} = \sqrt{1 - (2x)^2} = \cos \theta$ .

$$\begin{aligned} \int \sqrt{1 - 4x^2} dx &= \int \cos \theta \left( \frac{1}{2} \cos \theta \right) d\theta = \frac{1}{4} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{4} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{4} [\sin^{-1}(2x) + 2x \sqrt{1 - 4x^2}] + C \end{aligned}$$

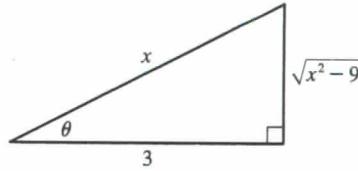


12.  $\int_0^1 x \sqrt{x^2 + 4} dx = \int_4^5 \sqrt{u} \left(\frac{1}{2} du\right)$  [ $u = x^2 + 4$ ,  $du = 2x dx$ ]  $= \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2}\right]_4^5 = \frac{1}{3}(5\sqrt{5} - 8)$

13. Let  $x = 3 \sec \theta$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ . Then

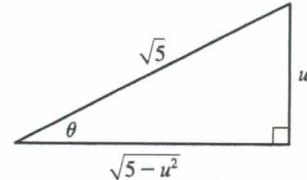
$$dx = 3 \sec \theta \tan \theta d\theta \text{ and } \sqrt{x^2 - 9} = 3 \tan \theta, \text{ so}$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - 9}}{x^3} dx &= \int \frac{3 \tan \theta}{27 \sec^3 \theta} 3 \sec \theta \tan \theta d\theta = \frac{1}{3} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta \\ &= \frac{1}{3} \int \sin^2 \theta d\theta = \frac{1}{3} \int \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{1}{6}\theta - \frac{1}{12} \sin 2\theta + C = \frac{1}{6}\theta - \frac{1}{6} \sin \theta \cos \theta + C \\ &= \frac{1}{6} \sec^{-1} \left( \frac{x}{3} \right) - \frac{1}{6} \frac{\sqrt{x^2 - 9}}{x} \frac{3}{x} + C = \frac{1}{6} \sec^{-1} \left( \frac{x}{3} \right) - \frac{\sqrt{x^2 - 9}}{2x^2} + C \end{aligned}$$



14. Let  $u = \sqrt{5} \sin \theta$ , so  $du = \sqrt{5} \cos \theta d\theta$ . Then

$$\begin{aligned} \int \frac{du}{u \sqrt{5 - u^2}} &= \int \frac{1}{\sqrt{5} \sin \theta \cdot \sqrt{5 \cos^2 \theta}} \sqrt{5} \cos \theta d\theta = \frac{1}{\sqrt{5}} \int \csc \theta d\theta \\ &= \frac{1}{\sqrt{5}} \ln |\csc \theta - \cot \theta| + C \quad [\text{by Exercise 7.2.39}] \\ &= \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5}}{u} - \frac{\sqrt{5 - u^2}}{u} \right| + C = \frac{1}{\sqrt{5}} \ln \left| \frac{\sqrt{5} - \sqrt{5 - u^2}}{u} \right| + C \end{aligned}$$



15. Let  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ ,  $x = 0 \Rightarrow \theta = 0$  and  $x = a \Rightarrow \theta = \frac{\pi}{2}$ . Then

$$\begin{aligned} \int_0^a x^2 \sqrt{a^2 - x^2} dx &= \int_0^{\pi/2} a^2 \sin^2 \theta (a \cos \theta) a \cos \theta d\theta = a^4 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = a^4 \int_0^{\pi/2} \left[ \frac{1}{2}(2 \sin \theta \cos \theta) \right]^2 d\theta \\ &= \frac{a^4}{4} \int_0^{\pi/2} \sin^2 2\theta d\theta = \frac{a^4}{4} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{a^4}{8} \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} \\ &= \frac{a^4}{8} \left[ \left( \frac{\pi}{2} - 0 \right) - 0 \right] = \frac{\pi}{16} a^4 \end{aligned}$$

16. Let  $x = \frac{1}{3} \sec \theta$ , so  $dx = \frac{1}{3} \sec \theta \tan \theta d\theta$ ,  $x = \sqrt{2}/3 \Rightarrow \theta = \frac{\pi}{4}$ ,  $x = \frac{2}{3} \Rightarrow \theta = \frac{\pi}{3}$ . Then

$$\begin{aligned} \int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5 \sqrt{9x^2 - 1}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3} \sec \theta \tan \theta d\theta}{\left(\frac{1}{3}\right)^5 \sec^5 \theta \tan \theta} = 3^4 \int_{\pi/4}^{\pi/3} \cos^4 \theta d\theta = 81 \int_{\pi/4}^{\pi/3} \left[ \frac{1}{2}(1 + \cos 2\theta) \right]^2 d\theta \\ &= \frac{81}{4} \int_{\pi/4}^{\pi/3} (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta = \frac{81}{4} \int_{\pi/4}^{\pi/3} [1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)] d\theta \\ &= \frac{81}{4} \int_{\pi/4}^{\pi/3} (\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta) d\theta = \frac{81}{4} \left[ \frac{3}{2}\theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_{\pi/4}^{\pi/3} \\ &= \frac{81}{4} \left[ \left( \frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{16} \right) - \left( \frac{3\pi}{8} + 1 + 0 \right) \right] = \frac{81}{4} \left( \frac{\pi}{8} + \frac{7}{16}\sqrt{3} - 1 \right) \end{aligned}$$

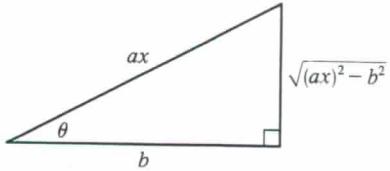
17. Let  $u = x^2 - 7$ , so  $du = 2x dx$ . Then  $\int \frac{x}{\sqrt{x^2 - 7}} dx = \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} \cdot 2\sqrt{u} + C = \sqrt{x^2 - 7} + C$ .

18. Let  $ax = b \sec \theta$ , so  $(ax)^2 = b^2 \sec^2 \theta \Rightarrow$

$$(ax)^2 - b^2 = b^2 \sec^2 \theta - b^2 = b^2(\sec^2 \theta - 1) = b^2 \tan^2 \theta.$$

So  $\sqrt{(ax)^2 - b^2} = b \tan \theta$ ,  $dx = \frac{b}{a} \sec \theta \tan \theta d\theta$ , and

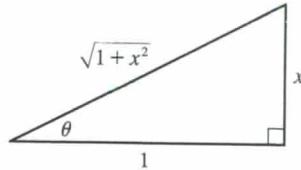
$$\begin{aligned} \int \frac{dx}{[(ax)^2 - b^2]^{3/2}} &= \int \frac{\frac{b}{a} \sec \theta \tan \theta}{b^3 \tan^3 \theta} d\theta = \frac{1}{ab^2} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{ab^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \frac{1}{ab^2} \int \csc \theta \cot \theta d\theta \\ &= -\frac{1}{ab^2} \csc \theta + C = -\frac{1}{ab^2} \frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2 \sqrt{(ax)^2 - b^2}} + C \end{aligned}$$



19. Let  $x = \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $dx = \sec^2 \theta d\theta$

and  $\sqrt{1+x^2} = \sec \theta$ , so

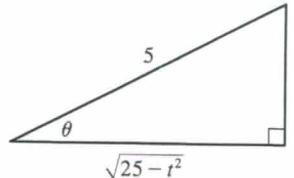
$$\begin{aligned} \int \frac{\sqrt{1+x^2}}{x} dx &= \int \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \int \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= \int (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= \ln |\csc \theta - \cot \theta| + \sec \theta + C \quad [\text{by Exercise 7.2.39}] \\ &= \ln \left| \frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right| + \frac{\sqrt{1+x^2}}{1} + C = \ln \left| \frac{\sqrt{1+x^2} - 1}{x} \right| + \sqrt{1+x^2} + C \end{aligned}$$



20. Let  $t = 5 \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dt = 5 \cos \theta d\theta$

and  $\sqrt{25-t^2} = 5 \cos \theta$ , so

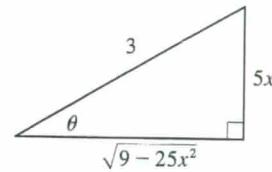
$$\begin{aligned} \int \frac{t}{\sqrt{25-t^2}} dt &= \int \frac{5 \sin \theta}{5 \cos \theta} 5 \cos \theta d\theta = 5 \int \sin \theta d\theta \\ &= -5 \cos \theta + C = -5 \cdot \frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C \end{aligned}$$



Or: Let  $u = 25 - t^2$ , so  $du = -2t dt$ .

21. Let  $x = \frac{3}{5} \sin \theta$ , so  $dx = \frac{3}{5} \cos \theta d\theta$ ,  $x = 0 \Rightarrow \theta = 0$ , and  $x = 0.6 \Rightarrow \theta = \frac{\pi}{2}$ . Then

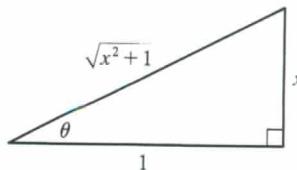
$$\begin{aligned} \int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}} dx &= \int_0^{\pi/2} \frac{\left(\frac{3}{5}\right)^2 \sin^2 \theta}{3 \cos \theta} \left(\frac{3}{5} \cos \theta d\theta\right) = \frac{9}{125} \int_0^{\pi/2} \sin^2 \theta d\theta \\ &= \frac{9}{125} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta) d\theta = \frac{9}{250} \left[\theta - \frac{1}{2} \sin 2\theta\right]_0^{\pi/2} \\ &= \frac{9}{250} \left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{9}{500} \pi \end{aligned}$$



22. Let  $x = \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $dx = \sec^2 \theta d\theta$ ,

$\sqrt{x^2+1} = \sec \theta$  and  $x = 0 \Rightarrow \theta = 0$ ,  $x = 1 \Rightarrow \theta = \frac{\pi}{4}$ , so

$$\begin{aligned} \int_0^1 \sqrt{x^2+1} dx &= \int_0^{\pi/4} \sec \theta \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^3 \theta d\theta \\ &= \frac{1}{2} \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} \quad [\text{by Example 7.2.8}] \\ &= \frac{1}{2} [\sqrt{2} \cdot 1 + \ln(1 + \sqrt{2}) - 0 - \ln(1 + 0)] = \frac{1}{2} [\sqrt{2} + \ln(1 + \sqrt{2})] \end{aligned}$$



23.  $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x - 2)^2 + 9$ . Let

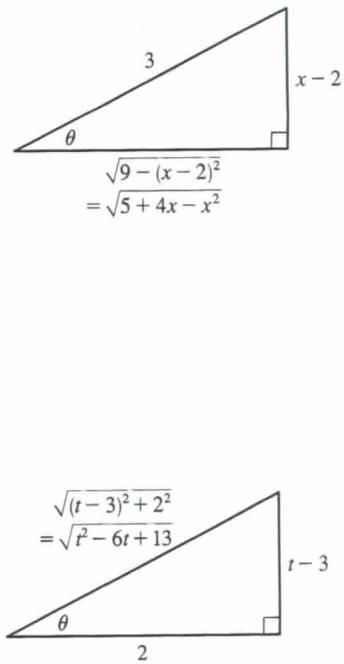
$x - 2 = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , so  $dx = 3 \cos \theta d\theta$ . Then

$$\begin{aligned}\int \sqrt{5 + 4x - x^2} dx &= \int \sqrt{9 - (x - 2)^2} dx = \int \sqrt{9 - 9 \sin^2 \theta} 3 \cos \theta d\theta \\&= \int \sqrt{9 \cos^2 \theta} 3 \cos \theta d\theta = \int 9 \cos^2 \theta d\theta \\&= \frac{9}{2} \int (1 + \cos 2\theta) d\theta = \frac{9}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\&= \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta + C = \frac{9}{2} \theta + \frac{9}{4} (2 \sin \theta \cos \theta) + C \\&= \frac{9}{2} \sin^{-1} \left( \frac{x - 2}{3} \right) + \frac{9}{2} \cdot \frac{x - 2}{3} \cdot \frac{\sqrt{5 + 4x - x^2}}{3} + C \\&= \frac{9}{2} \sin^{-1} \left( \frac{x - 2}{3} \right) + \frac{1}{2}(x - 2)\sqrt{5 + 4x - x^2} + C\end{aligned}$$

24.  $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4 = (t - 3)^2 + 2^2$ .

Let  $t - 3 = 2 \tan \theta$ , so  $dt = 2 \sec^2 \theta d\theta$ . Then

$$\begin{aligned}\int \frac{dt}{\sqrt{t^2 - 6t + 13}} &= \int \frac{1}{\sqrt{(2 \tan \theta)^2 + 2^2}} 2 \sec^2 \theta d\theta \\&= \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 \quad [\text{by Formula 7.2.1}] \\&= \ln \left| \frac{\sqrt{t^2 - 6t + 13}}{2} + \frac{t - 3}{2} \right| + C_1 \\&= \ln |\sqrt{t^2 - 6t + 13} + t - 3| + C \quad \text{where } C = C_1 - \ln 2\end{aligned}$$

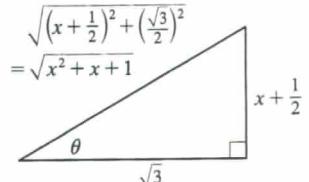


25.  $x^2 + x + 1 = (x^2 + x + \frac{1}{4}) + \frac{3}{4} = (x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2$ . Let

$x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$ , so  $dx = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$  and  $\sqrt{x^2 + x + 1} = \frac{\sqrt{3}}{2} \sec \theta$ .

Then

$$\begin{aligned}\int \frac{x}{\sqrt{x^2 + x + 1}} dx &= \int \frac{\frac{\sqrt{3}}{2} \tan \theta - \frac{1}{2}}{\frac{\sqrt{3}}{2} \sec \theta} \frac{\sqrt{3}}{2} \sec^2 \theta d\theta \\&= \int \left( \frac{\sqrt{3}}{2} \tan \theta - \frac{1}{2} \right) \sec \theta d\theta = \int \frac{\sqrt{3}}{2} \tan \theta \sec \theta d\theta - \int \frac{1}{2} \sec \theta d\theta \\&= \frac{\sqrt{3}}{2} \sec \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C_1 \\&= \sqrt{x^2 + x + 1} - \frac{1}{2} \ln \left| \frac{2}{\sqrt{3}} \sqrt{x^2 + x + 1} + \frac{2}{\sqrt{3}} (x + \frac{1}{2}) \right| + C_1 \\&= \sqrt{x^2 + x + 1} - \frac{1}{2} \ln \left| \frac{2}{\sqrt{3}} [\sqrt{x^2 + x + 1} + (x + \frac{1}{2})] \right| + C_1 \\&= \sqrt{x^2 + x + 1} - \frac{1}{2} \ln \frac{2}{\sqrt{3}} - \frac{1}{2} \ln (\sqrt{x^2 + x + 1} + x + \frac{1}{2}) + C_1 \\&= \sqrt{x^2 + x + 1} - \frac{1}{2} \ln (\sqrt{x^2 + x + 1} + x + \frac{1}{2}) + C, \quad \text{where } C = C_1 - \frac{1}{2} \ln \frac{2}{\sqrt{3}}\end{aligned}$$

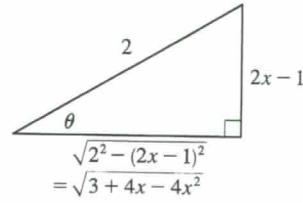


26.  $3 + 4x - 4x^2 = -(4x^2 - 4x + 1) + 4 = 2^2 - (2x - 1)^2$ .

Let  $2x - 1 = 2 \sin \theta$ , so  $2 dx = 2 \cos \theta d\theta$  and  $\sqrt{3 + 4x - 4x^2} = 2 \cos \theta$ .

Then

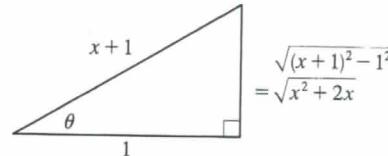
$$\begin{aligned} \int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}} dx &= \int \frac{\left[\frac{1}{2}(1 + 2 \sin \theta)\right]^2}{(2 \cos \theta)^3} \cos \theta d\theta \\ &= \frac{1}{32} \int \frac{1 + 4 \sin \theta + 4 \sin^2 \theta}{\cos^2 \theta} d\theta = \frac{1}{32} \int (\sec^2 \theta + 4 \tan \theta \sec \theta + 4 \tan^2 \theta) d\theta \\ &= \frac{1}{32} \int [\sec^2 \theta + 4 \tan \theta \sec \theta + 4(\sec^2 \theta - 1)] d\theta \\ &= \frac{1}{32} \int (5 \sec^2 \theta + 4 \tan \theta \sec \theta - 4) d\theta = \frac{1}{32} (5 \tan \theta + 4 \sec \theta - 4\theta) + C \\ &= \frac{1}{32} \left[ 5 \cdot \frac{2x - 1}{\sqrt{3 + 4x - 4x^2}} + 4 \cdot \frac{2}{\sqrt{3 + 4x - 4x^2}} - 4 \cdot \sin^{-1}\left(\frac{2x - 1}{2}\right) \right] + C \\ &= \frac{10x + 3}{32 \sqrt{3 + 4x - 4x^2}} - \frac{1}{8} \sin^{-1}\left(\frac{2x - 1}{2}\right) + C \end{aligned}$$



27.  $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$ . Let  $x + 1 = 1 \sec \theta$ ,

so  $dx = \sec \theta \tan \theta d\theta$  and  $\sqrt{x^2 + 2x} = \tan \theta$ . Then

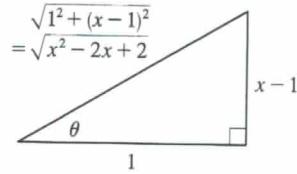
$$\begin{aligned} \int \sqrt{x^2 + 2x} dx &= \int \tan \theta (\sec \theta \tan \theta d\theta) = \int \tan^2 \theta \sec \theta d\theta \\ &= \int (\sec^2 \theta - 1) \sec \theta d\theta = \int \sec^3 \theta d\theta - \int \sec \theta d\theta \\ &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2}(x + 1)\sqrt{x^2 + 2x} - \frac{1}{2} \ln |x + 1 + \sqrt{x^2 + 2x}| + C \end{aligned}$$



28.  $x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x - 1)^2 + 1$ . Let  $x - 1 = 1 \tan \theta$ ,

so  $dx = \sec^2 \theta d\theta$  and  $\sqrt{x^2 - 2x + 2} = \sec \theta$ . Then

$$\begin{aligned} \int \frac{x^2 + 1}{(x^2 - 2x + 2)^2} dx &= \int \frac{(\tan \theta + 1)^2 + 1}{\sec^4 \theta} \sec^2 \theta d\theta \\ &= \int \frac{\tan^2 \theta + 2 \tan \theta + 2}{\sec^2 \theta} d\theta \\ &= \int (\sin^2 \theta + 2 \sin \theta \cos \theta + 2 \cos^2 \theta) d\theta = \int (1 + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\ &= \int [1 + 2 \sin \theta \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta = \int (\frac{3}{2} + 2 \sin \theta \cos \theta + \frac{1}{2} \cos 2\theta) d\theta \\ &= \frac{3}{2}\theta + \sin^2 \theta + \frac{1}{4} \sin 2\theta + C = \frac{3}{2}\theta + \sin^2 \theta + \frac{1}{2} \sin \theta \cos \theta + C \\ &= \frac{3}{2} \tan^{-1}\left(\frac{x - 1}{1}\right) + \frac{(x - 1)^2}{x^2 - 2x + 2} + \frac{1}{2} \frac{x - 1}{\sqrt{x^2 - 2x + 2}} \frac{1}{\sqrt{x^2 - 2x + 2}} + C \\ &= \frac{3}{2} \tan^{-1}(x - 1) + \frac{2(x^2 - 2x + 1) + x - 1}{2(x^2 - 2x + 2)} + C = \frac{3}{2} \tan^{-1}(x - 1) + \frac{2x^2 - 3x + 1}{2(x^2 - 2x + 2)} + C \end{aligned}$$



We can write the answer as

$$\begin{aligned} \frac{3}{2} \tan^{-1}(x-1) + \frac{(2x^2 - 4x + 4) + x - 3}{2(x^2 - 2x + 2)} + C &= \frac{3}{2} \tan^{-1}(x-1) + 1 + \frac{x-3}{2(x^2 - 2x + 2)} + C \\ &= \frac{3}{2} \tan^{-1}(x-1) + \frac{x-3}{2(x^2 - 2x + 2)} + C_1, \text{ where } C_1 = 1 + C \end{aligned}$$

29. Let  $u = x^2$ ,  $du = 2x dx$ . Then

$$\begin{aligned} \int x \sqrt{1-x^4} dx &= \int \sqrt{1-u^2} \left( \frac{1}{2} du \right) = \frac{1}{2} \int \cos \theta \cdot \cos \theta d\theta && \left[ \begin{array}{l} \text{where } u = \sin \theta, du = \cos \theta d\theta, \\ \text{and } \sqrt{1-u^2} = \cos \theta \end{array} \right] \\ &= \frac{1}{2} \int \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4} \theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + C \\ &= \frac{1}{4} \sin^{-1} u + \frac{1}{4} u \sqrt{1-u^2} + C = \frac{1}{4} \sin^{-1}(x^2) + \frac{1}{4} x^2 \sqrt{1-x^4} + C \end{aligned}$$

30. Let  $u = \sin t$ ,  $du = \cos t dt$ . Then

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}} dt &= \int_0^1 \frac{1}{\sqrt{1+u^2}} du = \int_0^{\pi/4} \frac{1}{\sec \theta} \sec^2 \theta d\theta && \left[ \begin{array}{l} \text{where } u = \tan \theta, du = \sec^2 \theta d\theta, \\ \text{and } \sqrt{1+u^2} = \sec \theta \end{array} \right] \\ &= \int_0^{\pi/4} \sec \theta d\theta = \left[ \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} && [\text{by (1) in Section 7.2}] \\ &= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1) \end{aligned}$$

31. (a) Let  $x = a \tan \theta$ , where  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then  $\sqrt{x^2 + a^2} = a \sec \theta$  and

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a \sec^2 \theta d\theta}{a \sec \theta} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C_1 = \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C_1 \\ &= \ln(x + \sqrt{x^2 + a^2}) + C \quad \text{where } C = C_1 - \ln |a| \end{aligned}$$

(b) Let  $x = a \sinh t$ , so that  $dx = a \cosh t dt$  and  $\sqrt{x^2 + a^2} = a \cosh t$ . Then

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \cosh t dt}{a \cosh t} = t + C = \sinh^{-1} \frac{x}{a} + C.$$

32. (a) Let  $x = a \tan \theta$ ,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ . Then

$$\begin{aligned} I &= \int \frac{x^2}{(x^2 + a^2)^{3/2}} dx = \int \frac{a^2 \tan^2 \theta}{a^3 \sec^3 \theta} a \sec^2 \theta d\theta = \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int \frac{\sec^2 \theta - 1}{\sec \theta} d\theta \\ &= \int (\sec \theta - \cos \theta) d\theta = \ln |\sec \theta + \tan \theta| - \sin \theta + C \\ &= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| - \frac{x}{\sqrt{x^2 + a^2}} + C = \ln(x + \sqrt{x^2 + a^2}) - \frac{x}{\sqrt{x^2 + a^2}} + C_1 \end{aligned}$$

(b) Let  $x = a \sinh t$ . Then

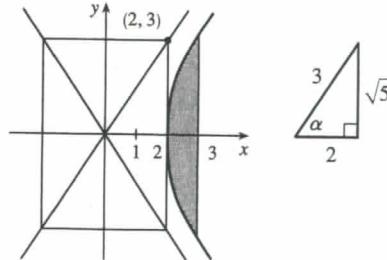
$$\begin{aligned} I &= \int \frac{a^2 \sinh^2 t}{a^3 \cosh^3 t} a \cosh t dt = \int \tanh^2 t dt = \int (1 - \operatorname{sech}^2 t) dt = t - \tanh t + C \\ &= \sinh^{-1} \frac{x}{a} - \frac{x}{\sqrt{a^2 + x^2}} + C \end{aligned}$$

33. The average value of  $f(x) = \sqrt{x^2 - 1}/x$  on the interval  $[1, 7]$  is

$$\begin{aligned} \frac{1}{7-1} \int_1^7 \frac{\sqrt{x^2 - 1}}{x} dx &= \frac{1}{6} \int_0^\alpha \frac{\tan \theta}{\sec \theta} \cdot \sec \theta \tan \theta d\theta && \left[ \text{where } x = \sec \theta, dx = \sec \theta \tan \theta d\theta, \right. \\ &= \frac{1}{6} \int_0^\alpha \tan^2 \theta d\theta = \frac{1}{6} \int_0^\alpha (\sec^2 \theta - 1) d\theta = \frac{1}{6} [\tan \theta - \theta]_0^\alpha \\ &= \frac{1}{6} (\tan \alpha - \alpha) = \frac{1}{6} (\sqrt{48} - \sec^{-1} 7) \end{aligned}$$

34.  $9x^2 - 4y^2 = 36 \Rightarrow y = \pm \frac{3}{2}\sqrt{x^2 - 4} \Rightarrow$

$$\begin{aligned} \text{area} &= 2 \int_2^3 \frac{3}{2} \sqrt{x^2 - 4} dx = 3 \int_2^3 \sqrt{x^2 - 4} dx \\ &= 3 \int_0^\alpha 2 \tan \theta 2 \sec \theta \tan \theta d\theta && \left[ \begin{array}{l} \text{where } x = 2 \sec \theta, \\ dx = 2 \sec \theta \tan \theta d\theta, \\ \alpha = \sec^{-1}(\frac{3}{2}) \end{array} \right] \\ &= 12 \int_0^\alpha (\sec^2 \theta - 1) \sec \theta d\theta = 12 \int_0^\alpha (\sec^3 \theta - \sec \theta) d\theta \\ &= 12 \left[ \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) - \ln |\sec \theta + \tan \theta| \right]_0^\alpha \\ &= 6 \left[ \sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| \right]_0^\alpha \\ &= 6 \left[ \frac{3\sqrt{5}}{4} - \ln \left( \frac{3}{2} + \frac{\sqrt{5}}{2} \right) \right] = \frac{9\sqrt{5}}{2} - 6 \ln \left( \frac{3+\sqrt{5}}{2} \right) \end{aligned}$$



35. Area of  $\triangle POQ = \frac{1}{2}(r \cos \theta)(r \sin \theta) = \frac{1}{2}r^2 \sin \theta \cos \theta$ . Area of region  $PQR = \int_{r \cos \theta}^r \sqrt{r^2 - x^2} dx$ .

Let  $x = r \cos u \Rightarrow dx = -r \sin u du$  for  $\theta \leq u \leq \frac{\pi}{2}$ . Then we obtain

$$\begin{aligned} \int \sqrt{r^2 - x^2} dx &= \int r \sin u (-r \sin u) du = -r^2 \int \sin^2 u du = -\frac{1}{2}r^2(u - \sin u \cos u) + C \\ &= -\frac{1}{2}r^2 \cos^{-1}(x/r) + \frac{1}{2}x \sqrt{r^2 - x^2} + C \end{aligned}$$

so

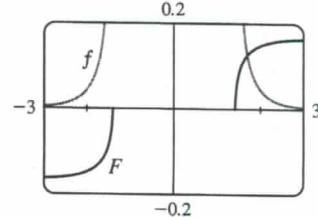
$$\begin{aligned} \text{area of region } PQR &= \frac{1}{2} \left[ -r^2 \cos^{-1}(x/r) + x \sqrt{r^2 - x^2} \right]_{r \cos \theta}^r \\ &= \frac{1}{2} [0 - (-r^2 \theta + r \cos \theta r \sin \theta)] = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \cos \theta \end{aligned}$$

and thus, (area of sector  $POR$ ) = (area of  $\triangle POQ$ ) + (area of region  $PQR$ ) =  $\frac{1}{2}r^2 \theta$ .

36. Let  $x = \sqrt{2} \sec \theta$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ , so  $dx = \sqrt{2} \sec \theta \tan \theta d\theta$ . Then

$$\begin{aligned} \int \frac{dx}{x^4 \sqrt{x^2 - 2}} &= \int \frac{\sqrt{2} \sec \theta \tan \theta d\theta}{4 \sec^4 \theta \sqrt{2} \tan \theta} \\ &= \frac{1}{4} \int \cos^3 \theta d\theta = \frac{1}{4} \int (1 - \sin^2 \theta) \cos \theta d\theta \\ &= \frac{1}{4} [\sin \theta - \frac{1}{3} \sin^3 \theta] + C && [\text{substitute } u = \sin \theta] \\ &= \frac{1}{4} \left[ \frac{\sqrt{x^2 - 2}}{x} - \frac{(x^2 - 2)^{3/2}}{3x^3} \right] + C \end{aligned}$$

From the graph, it appears that our answer is reasonable. [Notice that  $f(x)$  is large when  $F$  increases rapidly and small when  $F$  levels out.]



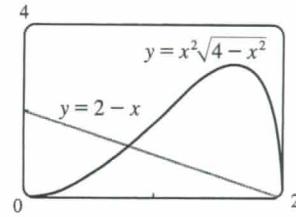
37. From the graph, it appears that the curve  $y = x^2 \sqrt{4 - x^2}$  and the line  $y = 2 - x$  intersect at about  $x = a \approx 0.81$  and  $x = 2$ , with  $x^2 \sqrt{4 - x^2} > 2 - x$  on  $(a, 2)$ . So the area bounded by the curve and the line is
- $$A \approx \int_a^2 [x^2 \sqrt{4 - x^2} - (2 - x)] dx = \int_a^2 x^2 \sqrt{4 - x^2} dx - [2x - \frac{1}{2}x^2]_a^2.$$

To evaluate the integral, we put  $x = 2 \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then

$$dx = 2 \cos \theta d\theta, x = 2 \Rightarrow \theta = \sin^{-1} 1 = \frac{\pi}{2}, \text{ and } x = a \Rightarrow \theta = \alpha = \sin^{-1}(a/2) \approx 0.416. \text{ So}$$

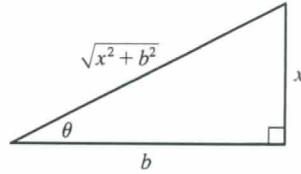
$$\begin{aligned} \int_a^2 x^2 \sqrt{4 - x^2} dx &\approx \int_{\alpha}^{\pi/2} 4 \sin^2 \theta (2 \cos \theta) (2 \cos \theta d\theta) = 4 \int_{\alpha}^{\pi/2} \sin^2 2\theta d\theta = 4 \int_{\alpha}^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta \\ &= 2[\theta - \frac{1}{4} \sin 4\theta]_{\alpha}^{\pi/2} = 2[(\frac{\pi}{2} - 0) - (\alpha - \frac{1}{4}(0.996))] \approx 2.81 \end{aligned}$$

Thus,  $A \approx 2.81 - [(2 \cdot 2 - \frac{1}{2} \cdot 2^2) - (2a - \frac{1}{2}a^2)] \approx 2.10$ .



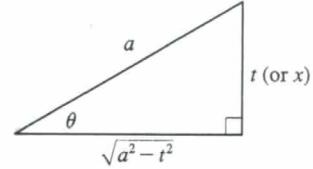
38. Let  $x = b \tan \theta$ , so that  $dx = b \sec^2 \theta d\theta$  and  $\sqrt{x^2 + b^2} = b \sec \theta$ .

$$\begin{aligned} E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}} dx = \frac{\lambda b}{4\pi\varepsilon_0} \int_{\theta_1}^{\theta_2} \frac{1}{(b \sec \theta)^3} b \sec^2 \theta d\theta \\ &= \frac{\lambda}{4\pi\varepsilon_0 b} \int_{\theta_1}^{\theta_2} \frac{1}{\sec \theta} d\theta = \frac{\lambda}{4\pi\varepsilon_0 b} \int_{\theta_1}^{\theta_2} \cos \theta d\theta = \frac{\lambda}{4\pi\varepsilon_0 b} [\sin \theta]_{\theta_1}^{\theta_2} \\ &= \frac{\lambda}{4\pi\varepsilon_0 b} \left[ \frac{x}{\sqrt{x^2 + b^2}} \right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b} \left( \frac{L-a}{\sqrt{(L-a)^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right) \end{aligned}$$



39. (a) Let  $t = a \sin \theta$ ,  $dt = a \cos \theta d\theta$ ,  $t = 0 \Rightarrow \theta = 0$  and  $t = x \Rightarrow \theta = \sin^{-1}(x/a)$ . Then

$$\begin{aligned} \int_0^x \sqrt{a^2 - t^2} dt &= \int_0^{\sin^{-1}(x/a)} a \cos \theta (a \cos \theta d\theta) \\ &= a^2 \int_0^{\sin^{-1}(x/a)} \cos^2 \theta d\theta = \frac{a^2}{2} \int_0^{\sin^{-1}(x/a)} (1 + \cos 2\theta) d\theta \\ &= \frac{a^2}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2} \left[ \theta + \sin \theta \cos \theta \right]_0^{\sin^{-1}(x/a)} \\ &= \frac{a^2}{2} \left[ \left( \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \right) - 0 \right] \\ &= \frac{1}{2} a^2 \sin^{-1}(x/a) + \frac{1}{2} x \sqrt{a^2 - x^2} \end{aligned}$$



- (b) The integral  $\int_0^x \sqrt{a^2 - t^2} dt$  represents the area under the curve  $y = \sqrt{a^2 - t^2}$  between the vertical lines  $t = 0$  and  $t = x$ .

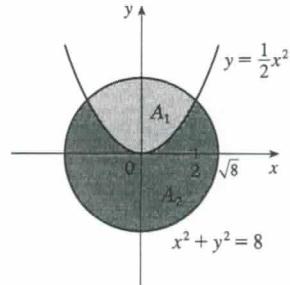
The figure shows that this area consists of a triangular region and a sector of the circle  $t^2 + y^2 = a^2$ . The triangular region has base  $x$  and height  $\sqrt{a^2 - x^2}$ , so its area is  $\frac{1}{2}x\sqrt{a^2 - x^2}$ . The sector has area  $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$ .

40. The curves intersect when  $x^2 + (\frac{1}{2}x^2)^2 = 8 \Leftrightarrow x^2 + \frac{1}{4}x^4 = 8 \Leftrightarrow x^4 + 4x^2 - 32 = 0 \Leftrightarrow$

$(x^2 + 8)(x^2 - 4) = 0 \Leftrightarrow x = \pm 2$ . The area inside the circle and above the parabola is given by

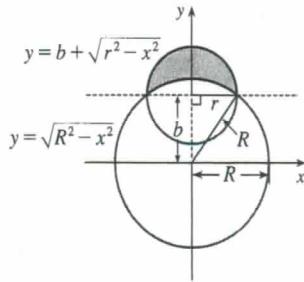
$$\begin{aligned}
 A_1 &= \int_{-2}^2 (\sqrt{8-x^2} - \frac{1}{2}x^2) dx = 2 \int_0^2 \sqrt{8-x^2} dx - 2 \int_0^2 \frac{1}{2}x^2 dx \\
 &= 2 \left[ \frac{1}{2}(8) \sin^{-1} \left( \frac{2}{\sqrt{8}} \right) + \frac{1}{2}(2) \sqrt{8-2^2} - \frac{1}{2} \left[ \frac{1}{3}x^3 \right]_0^2 \right] \quad [\text{by Exercise 39}] \\
 &= 8 \sin^{-1} \left( \frac{1}{\sqrt{2}} \right) + 2\sqrt{4} - \frac{8}{3} = 8 \left( \frac{\pi}{4} \right) + 4 - \frac{8}{3} = 2\pi + \frac{4}{3}
 \end{aligned}$$

Since the area of the disk is  $\pi(\sqrt{8})^2 = 8\pi$ , the area inside the circle and below the parabola is  $A_2 = 8\pi - (2\pi + \frac{4}{3}) = 6\pi - \frac{4}{3}$ .



41. Let the equation of the large circle be  $x^2 + y^2 = R^2$ . Then the equation of the small circle is  $x^2 + (y-b)^2 = r^2$ , where  $b = \sqrt{R^2 - r^2}$  is the distance between the centers of the circles. The desired area is

$$\begin{aligned}
 A &= \int_{-r}^r [(b + \sqrt{r^2 - x^2}) - \sqrt{R^2 - x^2}] dx \\
 &= 2 \int_0^r (b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}) dx \\
 &= 2 \int_0^r b dx + 2 \int_0^r \sqrt{r^2 - x^2} dx - 2 \int_0^r \sqrt{R^2 - x^2} dx
 \end{aligned}$$



The first integral is just  $2br = 2r\sqrt{R^2 - r^2}$ . The second integral represents the area of a quarter-circle of radius  $r$ , so its value is  $\frac{1}{4}\pi r^2$ . To evaluate the other integral, note that

$$\begin{aligned}
 \int \sqrt{a^2 - x^2} dx &= \int a^2 \cos^2 \theta d\theta \quad [x = a \sin \theta, dx = a \cos \theta d\theta] = \left( \frac{1}{2}a^2 \right) \int (1 + \cos 2\theta) d\theta \\
 &= \frac{1}{2}a^2 \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{2}a^2 (\theta + \sin \theta \cos \theta) + C \\
 &= \frac{a^2}{2} \arcsin \left( \frac{x}{a} \right) + \frac{a^2}{2} \left( \frac{x}{a} \right) \frac{\sqrt{a^2 - x^2}}{a} + C = \frac{a^2}{2} \arcsin \left( \frac{x}{a} \right) + \frac{x}{2} \sqrt{a^2 - x^2} + C
 \end{aligned}$$

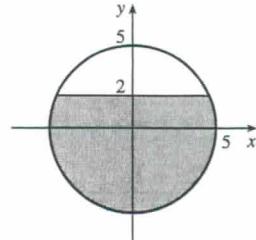
Thus, the desired area is

$$\begin{aligned}
 A &= 2r\sqrt{R^2 - r^2} + 2 \left( \frac{1}{4}\pi r^2 \right) - [R^2 \arcsin(x/R) + x\sqrt{R^2 - x^2}]_0^r \\
 &= 2r\sqrt{R^2 - r^2} + \frac{1}{2}\pi r^2 - [R^2 \arcsin(r/R) + r\sqrt{R^2 - r^2}] = r\sqrt{R^2 - r^2} + \frac{\pi}{2}r^2 - R^2 \arcsin(r/R)
 \end{aligned}$$

42. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is

$$\begin{aligned}
 A &= 2 \int_{-5}^2 \sqrt{25 - y^2} dy \\
 &= \left[ 25 \arcsin(y/5) + y\sqrt{25 - y^2} \right]_{-5}^2 \quad [\text{substitute } y = 5 \sin \theta] \\
 &= 25 \arcsin \frac{2}{5} + 2\sqrt{21} + \frac{25}{2}\pi \approx 58.72 \text{ ft}^2
 \end{aligned}$$

so the fraction of the total capacity in use is  $\frac{A}{\pi(5)^2} \approx \frac{58.72}{25\pi} \approx 0.748$  or 74.8%.



43. We use cylindrical shells and assume that  $R > r$ .  $x^2 = r^2 - (y - R)^2 \Rightarrow x = \pm\sqrt{r^2 - (y - R)^2}$ ,

so  $g(y) = 2\sqrt{r^2 - (y - R)^2}$  and

$$\begin{aligned} V &= \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y - R)^2} dy = \int_{-r}^r 4\pi(u + R)\sqrt{r^2 - u^2} du \quad [\text{where } u = y - R] \\ &= 4\pi \int_{-r}^r u \sqrt{r^2 - u^2} du + 4\pi R \int_{-r}^r \sqrt{r^2 - u^2} du \quad \left[ \begin{array}{l} \text{where } u = r \sin \theta, du = r \cos \theta d\theta \\ \text{in the second integral} \end{array} \right] \\ &= 4\pi \left[ -\frac{1}{3}(r^2 - u^2)^{3/2} \right]_{-r}^r + 4\pi R \int_{-\pi/2}^{\pi/2} r^2 \cos^2 \theta d\theta = -\frac{4\pi}{3}(0 - 0) + 4\pi R r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta \\ &= 2\pi R r^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta = 2\pi R r^2 [\theta + \frac{1}{2} \sin 2\theta]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2 \end{aligned}$$

*Another method:* Use washers instead of shells, so  $V = 8\pi R \int_0^r \sqrt{r^2 - y^2} dy$  as in Exercise 6.2.63(a), but evaluate the integral using  $y = r \sin \theta$ .

## 7.4 Integration of Rational Functions by Partial Fractions

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1. (a)  $\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1}$

(b)  $\frac{1}{x^3 + 2x^2 + x} = \frac{1}{x(x^2 + 2x + 1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

2. (a)  $\frac{x}{x^2 + x - 2} = \frac{x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

(b)  $\frac{x^2}{x^2 + x + 2} = \frac{(x^2 + x + 2) - (x + 2)}{x^2 + x + 2} = 1 - \frac{x+2}{x^2 + x + 2}$

Notice that  $x^2 + x + 2$  can't be factored because its discriminant is  $b^2 - 4ac = -7 < 0$ .

3. (a)  $\frac{x^4 + 1}{x^5 + 4x^3} = \frac{x^4 + 1}{x^3(x^2 + 4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 4}$

(b)  $\frac{1}{(x^2 - 9)^2} = \frac{1}{[(x+3)(x-3)]^2} = \frac{1}{(x+3)^2(x-3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2}$

4. (a)  $\frac{x^3}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{(x+1)(x+3)} = x - 4 + \frac{A}{x+1} + \frac{B}{x+3}$

(b)  $\frac{2x+1}{(x+1)^3(x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$

5. (a)  $\frac{x^4}{x^4 - 1} = \frac{(x^4 - 1) + 1}{x^4 - 1} = 1 + \frac{1}{x^4 - 1} \quad [\text{or use long division}] = 1 + \frac{1}{(x^2 - 1)(x^2 + 1)}$

$$= 1 + \frac{1}{(x-1)(x+1)(x^2+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(b)  $\frac{t^4 + t^2 + 1}{(t^2 + 1)(t^2 + 4)^2} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+4} + \frac{Et+F}{(t^2+4)^2}$