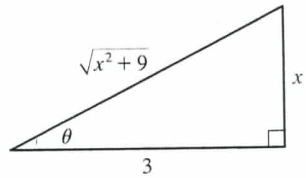
§7.3 Trig Subst. - HMWK SOLN's - pg. 1/2

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}$$
$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$

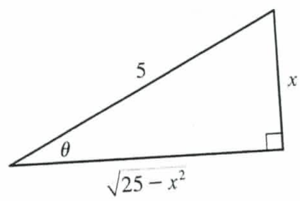


$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\,\sec\theta d\theta = 3^3\int \tan^2\theta\,\tan\theta\,\sec\theta\,d\theta$$
$$= 3^3\int(\sec^2\theta - 1)\,\tan\theta\,\sec\theta\,d\theta = 3^3\int(u^2-1)\,du \qquad [u = \sec\theta,\ du = \sec\theta\,\tan\theta\,d\theta]$$
$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$
$$= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \quad\text{or}\quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9} + C$$

5. Let $t = \sec\theta$, so $dt = \sec\theta\,\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$
$$= \int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$
$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

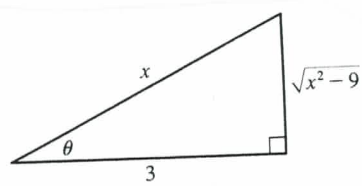
7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then



$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\,d\theta = \frac{1}{25}\int \csc^2\theta\,d\theta$$
$$= -\frac{1}{25}\cot\theta + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

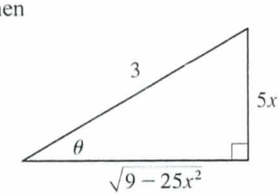
$dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$
$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$
$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then



$$\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2}\frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$
$$= \tfrac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$
$$= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2 \sin\theta\, \cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx$.

Let $x = r\cos u \;\Rightarrow\; dx = -r\sin u\, du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

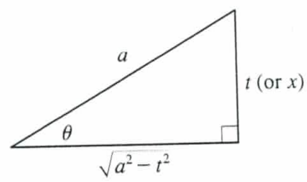
$$\begin{aligned}\int \sqrt{r^2 - x^2}\, dx &= \int r\sin u\,(-r\sin u)\, du = -r^2 \int \sin^2 u\, du = -\tfrac{1}{2}r^2(u - \sin u\, \cos u) + C \\ &= -\tfrac{1}{2}r^2 \cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2 - x^2} + C\end{aligned}$$

so

$$\begin{aligned}\text{area of region } PQR &= \tfrac{1}{2}\Big[-r^2\cos^{-1}(x/r) + x\sqrt{r^2 - x^2}\,\Big]_{r\cos\theta}^{r} \\ &= \tfrac{1}{2}\big[0 - (-r^2\theta + r\cos\theta\, r\sin\theta)\big] = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\, \cos\theta\end{aligned}$$

and thus, (area of sector POR) $=$ (area of $\triangle POQ$) $+$ (area of region PQR) $= \frac{1}{2}r^2\theta$.

39. (a) Let $t = a\sin\theta$, $dt = a\cos\theta\, d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$ and $t = x \;\Rightarrow\; \theta = \sin^{-1}(x/a)$. Then



$$\begin{aligned}\int_0^x \sqrt{a^2 - t^2}\, dt &= \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\, d\theta) \\ &= a^2 \int_0^{\sin^{-1}(x/a)} \cos^2\theta\, d\theta = \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)} (1 + \cos 2\theta)\, d\theta \\ &= \frac{a^2}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\Big[\theta + \sin\theta\, \cos\theta\Big]_0^{\sin^{-1}(x/a)} \\ &= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2 - x^2}}{a}\right) - 0\right] \\ &= \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}\end{aligned}$$

(b) The integral $\int_0^x \sqrt{a^2 - t^2}\, dt$ represents the area under the curve $y = \sqrt{a^2 - t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2 + y^2 = a^2$. The triangular region has base x and height $\sqrt{a^2 - x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2 - x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$.