Techniques of Integration - Trig. Substition

From Stewart, Calculus (ET) 6th ed § 7.3

7.3 EXERCISES

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx; \quad x = 3\sec\theta$

2. $\int x^3\sqrt{9-x^2}\,dx; \quad x = 3\sin\theta$

3. $\int \frac{x^3}{\sqrt{x^2+9}}\,dx; \quad x = 3\tan\theta$

4–30 Evaluate the integral.

4. $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$

5. $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$

6. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

7. $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

8. $\int \frac{x^3}{\sqrt{x^2+100}}\,dx$

9. $\int \frac{dx}{\sqrt{x^2+16}}$

10. $\int \frac{t^5}{\sqrt{t^2+2}}\,dt$

11. $\int \sqrt{1-4x^2}\,dx$

12. $\int_0^1 x\sqrt{x^2+4}\,dx$

13. $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

14. $\int \frac{du}{u\sqrt{5-u^2}}$

15. $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

16. $\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

17. $\int \frac{x}{\sqrt{x^2-7}}\,dx$

18. $\int \frac{dx}{[(ax)^2-b^2]^{3/2}}$

19. $\int \frac{\sqrt{1+x^2}}{x}\,dx$

20. $\int \frac{t}{\sqrt{25-t^2}}\,dt$

21. $\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$

22. $\int_0^1 \sqrt{x^2+1}\,dx$

23. $\int \sqrt{5+4x-x^2}\,dx$

24. $\int \frac{dt}{\sqrt{t^2-6t+13}}$

25. $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

26. $\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$

27. $\int \sqrt{x^2+2x}\,dx$

28. $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

29. $\int x\sqrt{1-x^4}\,dx$

30. $\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

31. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 3.11.3.

32. Evaluate

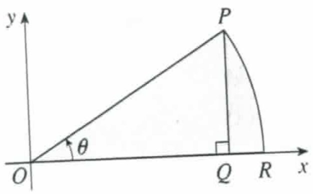
$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx$$

(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a\sinh t$.

33. Find the average value of $f(x) = \sqrt{x^2-1}/x$, $1 \leq x \leq 7$.

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

35. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



36. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

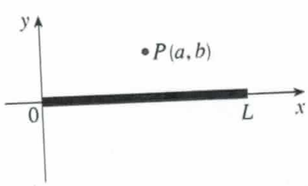
Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

37. Use a graph to approximate the roots of the equation $x^2\sqrt{4 - x^2} = 2 - x$. Then approximate the area bounded by the curve $y = x^2\sqrt{4 - x^2}$ and the line $y = 2 - x$.

38. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\,dx$$

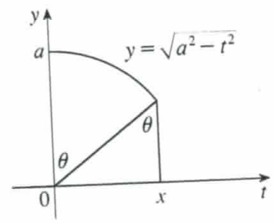
where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



39. (a) Use trigonometric substitution to verify that

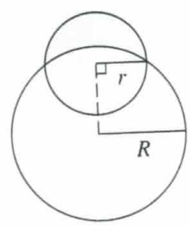
$$\int_0^x \sqrt{a^2 - t^2}\,dt = \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



40. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \leqslant 8$ into two parts. Find the areas of both parts.

41. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



42. A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?

43. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.