

(c) We divide the result from part (a) by  $I_{2n}$ . The inequalities are preserved since  $I_{2n}$  is positive:  $\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n}}{I_{2n}}$ .

Now from part (b), the left term is equal to  $\frac{2n+1}{2n+2}$ , so the expression becomes  $\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$ . Now

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} = \lim_{n \rightarrow \infty} 1 = 1, \text{ so by the Squeeze Theorem, } \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

(d) We substitute the results from Exercises 45 and 46 into the result from part (c):

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n \rightarrow \infty} \frac{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\pi}{2 \cdot 4 \cdot 6 \cdots (2n)}} = \lim_{n \rightarrow \infty} \left[ \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right] \left[ \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \left( \frac{2}{\pi} \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi} \quad [\text{rearrange terms}] \end{aligned}$$

Multiplying both sides by  $\frac{\pi}{2}$  gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

(e) The area of the  $k$ th rectangle is  $k$ . At the  $2n$ th step, the area is increased from  $2n-1$  to  $2n$  by multiplying the width by

$\frac{2n}{2n-1}$ , and at the  $(2n+1)$ th step, the area is increased from  $2n$  to  $2n+1$  by multiplying the height by  $\frac{2n+1}{2n}$ . These

two steps multiply the ratio of width to height by  $\frac{2n}{2n-1}$  and  $\frac{1}{(2n+1)/(2n)} = \frac{2n}{2n+1}$  respectively. So, by part (d), the

limiting ratio is  $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{\pi}{2}$ .

## 7.2 Trigonometric Integrals

The symbols  $\stackrel{s}{=}$  and  $\stackrel{c}{=}$  indicate the use of the substitutions  $\{u = \sin x, du = \cos x dx\}$  and  $\{u = \cos x, du = -\sin x dx\}$ , respectively.

$$\begin{aligned} 1. \int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx \stackrel{c}{=} \int (1 - u^2) u^2 (-du) \\ &= \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \end{aligned}$$

$$\begin{aligned} 2. \int \sin^6 x \cos^3 x dx &= \int \sin^6 x \cos^2 x \cos x dx = \int \sin^6 x (1 - \sin^2 x) \cos x dx \stackrel{s}{=} \int u^6 (1 - u^2) du \\ &= \int (u^6 - u^8) du = \frac{1}{7} u^7 - \frac{1}{9} u^9 + C = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C \end{aligned}$$

$$\begin{aligned} 3. \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x dx &= \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x dx = \int_{\pi/2}^{3\pi/4} \sin^5 x (1 - \sin^2 x) \cos x dx \stackrel{s}{=} \int_1^{\sqrt{2}/2} u^5 (1 - u^2) du \\ &= \int_1^{\sqrt{2}/2} (u^5 - u^7) du = \left[ \frac{1}{6} u^6 - \frac{1}{8} u^8 \right]_1^{\sqrt{2}/2} = \left( \frac{1}{6} \left( \frac{1}{2} \right)^6 - \frac{1}{8} \left( \frac{1}{2} \right)^8 \right) - \left( \frac{1}{6} - \frac{1}{8} \right) = -\frac{11}{384} \end{aligned}$$

$$\begin{aligned} 4. \int_0^{\pi/2} \cos^5 x dx &= \int_0^{\pi/2} (\cos^2 x)^2 \cos x dx = \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x dx \stackrel{s}{=} \int_0^1 (1 - u^2)^2 du \\ &= \int_0^1 (1 - 2u^2 + u^4) du = \left[ u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right]_0^1 = \left( 1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{8}{15} \end{aligned}$$

5. Let  $y = \pi x$ , so  $dy = \pi dx$  and

$$\begin{aligned} \int \sin^2(\pi x) \cos^5(\pi x) dx &= \frac{1}{\pi} \int \sin^2 y \cos^5 y dy = \frac{1}{\pi} \int \sin^2 y \cos^4 y \cos y dy \\ &= \frac{1}{\pi} \int \sin^2 y (1 - \sin^2 y)^2 \cos y dy \stackrel{u = \sin y}{=} \frac{1}{\pi} \int u^2 (1 - u^2)^2 du = \frac{1}{\pi} \int (u^2 - 2u^4 + u^6) du \\ &= \frac{1}{\pi} \left( \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 \right) + C = \frac{1}{3\pi} \sin^3 y - \frac{2}{5\pi} \sin^5 y + \frac{1}{7\pi} \sin^7 y + C \\ &= \frac{1}{3\pi} \sin^3(\pi x) - \frac{2}{5\pi} \sin^5(\pi x) + \frac{1}{7\pi} \sin^7(\pi x) + C \end{aligned}$$

6. Let  $y = \sqrt{x}$ , so that  $dy = \frac{1}{2\sqrt{x}} dx$  and  $dx = 2y dy$ . Then

$$\begin{aligned} \int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx &= \int \frac{\sin^3 y}{y} (2y dy) = 2 \int \sin^3 y \sin y dy = 2 \int (1 - \cos^2 y) \sin y dy \\ &\stackrel{u = \sin y}{=} 2 \int (1 - u^2)(-du) = 2 \int (u^2 - 1) du = 2 \left( \frac{1}{3}u^3 - u \right) + C = \frac{2}{3} \cos^3 y - 2 \cos y + C \\ &= \frac{2}{3} \cos^3(\sqrt{x}) - 2 \cos \sqrt{x} + C \end{aligned}$$

7.  $\int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta$  [half-angle identity]

$$\begin{aligned} &= \frac{1}{2} [\theta + \frac{1}{2} \sin 2\theta]_0^{\pi/2} = \frac{1}{2} \left[ \left( \frac{\pi}{2} + 0 \right) - (0 + 0) \right] = \frac{\pi}{4} \end{aligned}$$

$$8. \int_0^{\pi/2} \sin^2(2\theta) d\theta = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{1}{2} [\theta - \frac{1}{4} \sin 4\theta]_0^{\pi/2} = \frac{1}{2} \left[ \left( \frac{\pi}{2} - 0 \right) - (0 - 0) \right] = \frac{\pi}{4}$$

$$\begin{aligned} 9. \int_0^{\pi} \sin^4(3t) dt &= \int_0^{\pi} [\sin^2(3t)]^2 dt = \int_0^{\pi} \left[ \frac{1}{2}(1 - \cos 6t) \right]^2 dt = \frac{1}{4} \int_0^{\pi} (1 - 2 \cos 6t + \cos^2 6t) dt \\ &= \frac{1}{4} \int_0^{\pi} [1 - 2 \cos 6t + \frac{1}{2}(1 + \cos 12t)] dt = \frac{1}{4} \int_0^{\pi} \left( \frac{3}{2} - 2 \cos 6t + \frac{1}{2} \cos 12t \right) dt \\ &= \frac{1}{4} \left[ \frac{3}{2}t - \frac{1}{3} \sin 6t + \frac{1}{24} \sin 12t \right]_0^{\pi} = \frac{1}{4} \left[ \left( \frac{3\pi}{2} - 0 + 0 \right) - (0 - 0 + 0) \right] = \frac{3\pi}{8} \end{aligned}$$

$$\begin{aligned} 10. \int_0^{\pi} \cos^6 \theta d\theta &= \int_0^{\pi} (\cos^2 \theta)^3 d\theta = \int_0^{\pi} \left[ \frac{1}{2}(1 + \cos 2\theta) \right]^3 d\theta = \frac{1}{8} \int_0^{\pi} (1 + 3 \cos 2\theta + 3 \cos^2 2\theta + \cos^3 2\theta) d\theta \\ &= \frac{1}{8} [\theta + \frac{3}{2} \sin 2\theta]_0^{\pi} + \frac{1}{8} \int_0^{\pi} [\frac{3}{2}(1 + \cos 4\theta)] d\theta + \frac{1}{8} \int_0^{\pi} [(1 - \sin^2 2\theta) \cos 2\theta] d\theta \\ &= \frac{1}{8}\pi + \frac{3}{16} [\theta + \frac{1}{4} \sin 4\theta]_0^{\pi} + \frac{1}{8} \int_0^{\pi} (1 - u^2)(\frac{1}{2}du) \quad [u = \sin 2\theta, du = 2 \cos 2\theta d\theta] \\ &= \frac{\pi}{8} + \frac{3\pi}{16} + 0 = \frac{5\pi}{16} \end{aligned}$$

$$11. \int (1 + \cos \theta)^2 d\theta = \int (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \theta + 2 \sin \theta + \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$\begin{aligned} &= \theta + 2 \sin \theta + \frac{1}{2}\theta + \frac{1}{4} \sin 2\theta + C = \frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta + C \end{aligned}$$

$$12. \text{Let } u = x, dv = \cos^2 x dx \Rightarrow du = dx, v = \int \cos^2 x dx = \int \frac{1}{2}(1 + \cos 2x) dx = \frac{1}{2}x + \frac{1}{4} \sin 2x, \text{ so}$$

$$\begin{aligned} \int x \cos^2 x dx &= x \left( \frac{1}{2}x + \frac{1}{4} \sin 2x \right) - \int \left( \frac{1}{2}x + \frac{1}{4} \sin 2x \right) dx = \frac{1}{2}x^2 + \frac{1}{4}x \sin 2x - \frac{1}{4}x^2 + \frac{1}{8} \cos 2x + C \\ &= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x + \frac{1}{8} \cos 2x + C \end{aligned}$$

$$\begin{aligned} 13. \int_0^{\pi/2} \sin^2 x \cos^2 x dx &= \int_0^{\pi/2} \frac{1}{4}(4 \sin^2 x \cos^2 x) dx = \int_0^{\pi/2} \frac{1}{4}(2 \sin x \cos x)^2 dx = \frac{1}{4} \int_0^{\pi/2} \sin^2 2x dx \\ &= \frac{1}{4} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) dx = \frac{1}{8} \int_0^{\pi/2} (1 - \cos 4x) dx = \frac{1}{8} [x - \frac{1}{4} \sin 4x]_0^{\pi/2} = \frac{1}{8} \left( \frac{\pi}{2} \right) = \frac{\pi}{16} \end{aligned}$$

$$\begin{aligned}
 14. \int_0^\pi \sin^2 t \cos^4 t dt &= \frac{1}{4} \int_0^\pi (4 \sin^2 t \cos^2 t) \cos^2 t dt = \frac{1}{4} \int_0^\pi (2 \sin t \cos t)^2 \frac{1}{2} (1 + \cos 2t) dt \\
 &= \frac{1}{8} \int_0^\pi (\sin 2t)^2 (1 + \cos 2t) dt = \frac{1}{8} \int_0^\pi (\sin^2 2t + \sin^2 2t \cos 2t) dt \\
 &= \frac{1}{8} \int_0^\pi \sin^2 2t dt + \frac{1}{8} \int_0^\pi \sin^2 2t \cos 2t dt = \frac{1}{8} \int_0^\pi \frac{1}{2} (1 - \cos 4t) dt + \frac{1}{8} \left[ \frac{1}{3} \cdot \frac{1}{2} \sin^3 2t \right]_0^\pi \\
 &= \frac{1}{16} \left[ t - \frac{1}{4} \sin 4t \right]_0^\pi + \frac{1}{8} (0 - 0) = \frac{1}{16} [(\pi - 0) - 0] = \frac{\pi}{16}
 \end{aligned}$$

$$\begin{aligned}
 15. \int \frac{\cos^5 \alpha}{\sqrt{\sin \alpha}} d\alpha &= \int \frac{\cos^4 \alpha}{\sqrt{\sin \alpha}} \cos \alpha d\alpha = \int \frac{(1 - \sin^2 \alpha)^2}{\sqrt{\sin \alpha}} \cos \alpha d\alpha \stackrel{u}{=} \int \frac{(1 - u^2)^2}{\sqrt{u}} du \\
 &= \int \frac{1 - 2u^2 + u^4}{u^{1/2}} du = \int (u^{-1/2} - 2u^{3/2} + u^{7/2}) du = 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C \\
 &= \frac{2}{45}u^{1/2}(45 - 18u^2 + 5u^4) + C = \frac{2}{45}\sqrt{\sin \alpha}(45 - 18 \sin^2 \alpha + 5 \sin^4 \alpha) + C
 \end{aligned}$$

16. Let  $u = \sin \theta$ . Then  $du = \cos \theta d\theta$  and

$$\begin{aligned}
 \int \cos \theta \cos^5(\sin \theta) d\theta &= \int \cos^5 u du = \int (\cos^2 u)^2 \cos u du = \int (1 - \sin^2 u)^2 \cos u du \\
 &= \int (1 - 2 \sin^2 u + \sin^4 u) \cos u du = I
 \end{aligned}$$

Now let  $x = \sin u$ . Then  $dx = \cos u du$  and

$$\begin{aligned}
 I &= \int (1 - 2x^2 + x^4) dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C = \sin u - \frac{2}{3}\sin^3 u + \frac{1}{5}\sin^5 u + C \\
 &= \sin(\sin \theta) - \frac{2}{3}\sin^3(\sin \theta) + \frac{1}{5}\sin^5(\sin \theta) + C
 \end{aligned}$$

$$\begin{aligned}
 17. \int \cos^2 x \tan^3 x dx &= \int \frac{\sin^3 x}{\cos x} dx \stackrel{u}{=} \int \frac{(1 - u^2)(-du)}{u} = \int \left[ \frac{-1}{u} + u \right] du \\
 &= -\ln|u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C
 \end{aligned}$$

$$\begin{aligned}
 18. \int \cot^5 \theta \sin^4 \theta d\theta &= \int \frac{\cos^5 \theta}{\sin^5 \theta} \sin^4 \theta d\theta = \int \frac{\cos^5 \theta}{\sin \theta} d\theta \stackrel{u}{=} \int \frac{\cos^4 \theta}{\sin \theta} \cos \theta d\theta = \int \frac{(1 - \sin^2 \theta)^2}{\sin \theta} \cos \theta d\theta \\
 &\stackrel{u}{=} \int \frac{(1 - u^2)^2}{u} du = \int \frac{1 - 2u^2 + u^4}{u} du = \int \left( \frac{1}{u} - 2u + u^3 \right) du \\
 &= \ln|u| - u^2 + \frac{1}{4}u^4 + C = \ln|\sin \theta| - \sin^2 \theta + \frac{1}{4}\sin^4 \theta + C
 \end{aligned}$$

$$\begin{aligned}
 19. \int \frac{\cos x + \sin 2x}{\sin x} dx &= \int \frac{\cos x + 2 \sin x \cos x}{\sin x} dx = \int \frac{\cos x}{\sin x} dx + \int 2 \cos x dx \stackrel{u}{=} \int \frac{1}{u} du + 2 \sin x \\
 &= \ln|u| + 2 \sin x + C = \ln|\sin x| + 2 \sin x + C
 \end{aligned}$$

Or: Use the formula  $\int \cot x dx = \ln|\sin x| + C$ .

$$20. \int \cos^2 x \sin 2x dx = 2 \int \cos^3 x \sin x dx \stackrel{u}{=} -2 \int u^3 du = -\frac{1}{2}u^4 + C = -\frac{1}{2}\cos^4 x + C$$

$$21. \text{Let } u = \tan x, du = \sec^2 x dx. \text{ Then } \int \sec^2 x \tan x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

Or: Let  $v = \sec x, dv = \sec x \tan x dx$ . Then  $\int \sec^2 x \tan x dx = \int v dv = \frac{1}{2}v^2 + C = \frac{1}{2}\sec^2 x + C$ .

$$\begin{aligned}
 22. \int_0^{\pi/2} \sec^4(t/2) dt &= \int_0^{\pi/4} \sec^4 x (2 dx) \quad [x = t/2, dx = \frac{1}{2}dt] \quad = 2 \int_0^{\pi/4} \sec^2 x (1 + \tan^2 x) dx \\
 &= 2 \int_0^1 (1 + u^2) du \quad [u = \tan x, du = \sec^2 x dx] \quad = 2[u + \frac{1}{3}u^3]_0^1 = 2(1 + \frac{1}{3}) = \frac{8}{3}
 \end{aligned}$$

23.  $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

24.  $\int (\tan^2 x + \tan^4 x) dx = \int \tan^2 x (1 + \tan^2 x) dx = \int \tan^2 x \sec^2 x dx = \int u^2 du$  [ $u = \tan x, du = \sec^2 x dx$ ]  
 $= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

25.  $\int \sec^6 t dt = \int \sec^4 t \cdot \sec^2 t dt = \int (\tan^2 t + 1)^2 \sec^2 t dt = \int (u^2 + 1)^2 du$  [ $u = \tan t, du = \sec^2 t dt$ ]

$= \int (u^4 + 2u^2 + 1) du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C = \frac{1}{5}\tan^5 t + \frac{2}{3}\tan^3 t + \tan t + C$

26.  $\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta d\theta = \int_0^{\pi/4} (\tan^2 \theta + 1) \tan^4 \theta \sec^2 \theta d\theta = \int_0^1 (u^2 + 1) u^4 du$  [ $u = \tan \theta, du = \sec^2 \theta d\theta$ ]  
 $= \int_0^1 (u^6 + u^4) du = [\frac{1}{7}u^7 + \frac{1}{5}u^5]_0^1 = \frac{1}{7} + \frac{1}{5} = \frac{12}{35}$

27.  $\int_0^{\pi/3} \tan^5 x \sec^4 x dx = \int_0^{\pi/3} \tan^5 x (\tan^2 x + 1) \sec^2 x dx = \int_0^{\sqrt{3}} u^5 (u^2 + 1) du$  [ $u = \tan x, du = \sec^2 x dx$ ]  
 $= \int_0^{\sqrt{3}} (u^7 + u^5) du = [\frac{1}{8}u^8 + \frac{1}{6}u^6]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

Alternate solution:

$$\begin{aligned} \int_0^{\pi/3} \tan^5 x \sec^4 x dx &= \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x dx \\ &= \int_1^2 (u^2 - 1)^2 u^3 du \quad [u = \sec x, du = \sec x \tan x dx] = \int_1^2 (u^4 - 2u^2 + 1) u^3 du \\ &= \int_1^2 (u^7 - 2u^5 + u^3) du = [\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4]_1^2 = (32 - \frac{64}{3} + 4) - (\frac{1}{8} - \frac{1}{3} + \frac{1}{4}) = \frac{117}{8} \end{aligned}$$

28.  $\int \tan^3(2x) \sec^5(2x) dx = \int \tan^2(2x) \sec^4(2x) \cdot \sec(2x) \tan(2x) dx$   
 $= \int (u^2 - 1) u^4 (\frac{1}{2} du) \quad [u = \sec(2x), du = 2 \sec(2x) \tan(2x) dx]$   
 $= \frac{1}{2} \int (u^6 - u^4) du = \frac{1}{14}u^7 - \frac{1}{10}u^5 + C = \frac{1}{14}\sec^7(2x) - \frac{1}{10}\sec^5(2x) + C$

29.  $\int \tan^3 x \sec x dx = \int \tan^2 x \sec x \tan x dx = \int (\sec^2 x - 1) \sec x \tan x dx$   
 $= \int (u^2 - 1) du \quad [u = \sec x, du = \sec x \tan x dx] = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

30.  $\int_0^{\pi/3} \tan^5 x \sec^6 x dx = \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x dx = \int_0^{\pi/3} \tan^5 x (1 + \tan^2 x)^2 \sec^2 x dx$   
 $= \int_0^{\sqrt{3}} u^5 (1 + u^2)^2 du \quad [u = \tan x, du = \sec^2 x dx] = \int_0^{\sqrt{3}} u^5 (1 + 2u^2 + u^4) du$   
 $= \int_0^{\sqrt{3}} (u^5 + 2u^7 + u^9) du = [\frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10}]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}$

Alternate solution:

$$\begin{aligned} \int_0^{\pi/3} \tan^5 x \sec^6 x dx &= \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x dx \\ &= \int_1^2 (u^2 - 1)^2 u^5 du \quad [u = \sec x, du = \sec x \tan x dx] \\ &= \int_1^2 (u^4 - 2u^2 + 1) u^5 du = \int_1^2 (u^9 - 2u^7 + u^5) du \\ &= [\frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6]_1^2 = (\frac{512}{5} - 64 + \frac{32}{3}) - (\frac{1}{10} - \frac{1}{4} + \frac{1}{6}) = \frac{981}{20} \end{aligned}$$

31.  $\int \tan^5 x dx = \int (\sec^2 x - 1)^2 \tan x dx = \int \sec^4 x \tan x dx - 2 \int \sec^2 x \tan x dx + \int \tan x dx$   
 $= \int \sec^3 x \sec x \tan x dx - 2 \int \tan x \sec^2 x dx + \int \tan x dx$   
 $= \frac{1}{4}\sec^4 x - \tan^2 x + \ln |\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln |\sec x| + C]$

$$\begin{aligned}
 32. \int \tan^6 ay dy &= \int \tan^4 ay (\sec^2 ay - 1) dy = \int \tan^4 ay \sec^2 ay dy - \int \tan^4 ay dy \\
 &= \frac{1}{5a} \tan^5 ay - \int \tan^2 ay (\sec^2 ay - 1) dy = \frac{1}{5a} \tan^5 ay - \int \tan^2 ay \sec^2 ay dy + \int (\sec^2 ay - 1) dy \\
 &= \frac{1}{5a} \tan^5 ay - \frac{1}{3a} \tan^3 ay + \frac{1}{a} \tan ay - y + C
 \end{aligned}$$

$$\begin{aligned}
 33. \int \frac{\tan^3 \theta}{\cos^4 \theta} d\theta &= \int \tan^3 \theta \sec^4 \theta d\theta = \int \tan^3 \theta \cdot (\tan^2 \theta + 1) \cdot \sec^2 \theta d\theta \\
 &= \int u^3(u^2 + 1) du \quad [u = \tan \theta, du = \sec^2 \theta d\theta] \\
 &= \int (u^5 + u^3) du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6 \theta + \frac{1}{4}\tan^4 \theta + C
 \end{aligned}$$

$$\begin{aligned}
 34. \int \tan^2 x \sec x dx &= \int (\sec^2 x - 1) \sec x dx = \int \sec^3 x dx - \int \sec x dx \\
 &= \frac{1}{2}(\sec x \tan x + \ln |\sec x + \tan x|) - \ln |\sec x + \tan x| + C \quad [\text{by Example 8 and (1)}] \\
 &= \frac{1}{2}(\sec x \tan x - \ln |\sec x + \tan x|) + C
 \end{aligned}$$

35. Let  $u = x, dv = \sec x \tan x dx \Rightarrow du = dx, v = \sec x$ . Then

$$\int x \sec x \tan x dx = x \sec x - \int \sec x dx = x \sec x - \ln |\sec x + \tan x| + C.$$

$$\begin{aligned}
 36. \int \frac{\sin \phi}{\cos^3 \phi} d\phi &= \int \frac{\sin \phi}{\cos \phi} \cdot \frac{1}{\cos^2 \phi} d\phi = \int \tan \phi \sec^2 \phi d\phi = \int u du \quad [u = \tan \phi, du = \sec^2 \phi d\phi] \\
 &= \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 \phi + C
 \end{aligned}$$

*Alternate solution:* Let  $u = \cos \phi$  to get  $\frac{1}{2}\sec^2 \phi + C$ .

$$37. \int_{\pi/6}^{\pi/2} \cot^2 x dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1) dx = [-\cot x - x]_{\pi/6}^{\pi/2} = (0 - \frac{\pi}{2}) - (-\sqrt{3} - \frac{\pi}{6}) = \sqrt{3} - \frac{\pi}{3}$$

$$\begin{aligned}
 38. \int_{\pi/4}^{\pi/2} \cot^3 x dx &= \int_{\pi/4}^{\pi/2} \cot x (\csc^2 x - 1) dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx \\
 &= \left[ -\frac{1}{2}\cot^2 x - \ln |\sin x| \right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[ -\frac{1}{2} - \ln \frac{1}{\sqrt{2}} \right] = \frac{1}{2} + \ln \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2)
 \end{aligned}$$

$$\begin{aligned}
 39. \int \cot^3 \alpha \csc^3 \alpha d\alpha &= \int \cot^2 \alpha \csc^2 \alpha \cdot \csc \alpha \cot \alpha d\alpha = \int (\csc^2 \alpha - 1) \csc^2 \alpha \cdot \csc \alpha \cot \alpha d\alpha \\
 &= \int (u^2 - 1)u^2 \cdot (-du) \quad [u = \csc \alpha, du = -\csc \alpha \cot \alpha d\alpha] \\
 &= \int (u^2 - u^4) du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3 \alpha - \frac{1}{5}\csc^5 \alpha + C
 \end{aligned}$$

$$\begin{aligned}
 40. \int \csc^4 x \cot^6 x dx &= \int \cot^6 x (\cot^2 x + 1) \csc^2 x dx = \int u^6(u^2 + 1) \cdot (-du) \quad [u = \cot x, du = -\csc^2 x dx] \\
 &= \int u^6(u^2 + 1) \cdot (-du) \quad [u = \cot x, du = -\csc^2 x dx] \\
 &= \int (-u^8 - u^6) du = -\frac{1}{9}u^9 - \frac{1}{7}u^7 + C = -\frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C
 \end{aligned}$$

$$41. I = \int \csc x dx = \int \frac{\csc x (\csc x - \cot x)}{\csc x - \cot x} dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} dx. \text{ Let } u = \csc x - \cot x \Rightarrow \\
 du = (-\csc x \cot x + \csc^2 x) dx. \text{ Then } I = \int du/u = \ln |u| = \ln |\csc x - \cot x| + C.$$

42. Let  $u = \csc x$ ,  $dv = \csc^2 x dx$ . Then  $du = -\csc x \cot x dx$ ,  $v = -\cot x \Rightarrow$

$$\begin{aligned}\int \csc^3 x dx &= -\csc x \cot x - \int \csc x \cot^2 x dx = -\csc x \cot x - \int \csc x (\csc^2 x - 1) dx \\ &= -\csc x \cot x + \int \csc x dx - \int \csc^3 x dx\end{aligned}$$

Solving for  $\int \csc^3 x dx$  and using Exercise 41, we get

$$\int \csc^3 x dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \int \csc x dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| + C. \text{ Thus,}$$

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \csc^3 x dx &= \left[ -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/3} \\ &= -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{2} \ln \left| \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right| + \frac{1}{2} \cdot 2 \cdot \sqrt{3} - \frac{1}{2} \ln |2 - \sqrt{3}| \\ &= -\frac{1}{3} + \sqrt{3} + \frac{1}{2} \ln \frac{1}{\sqrt{3}} - \frac{1}{2} \ln (2 - \sqrt{3}) \approx 1.7825\end{aligned}$$

$$\begin{aligned}43. \int \sin 8x \cos 5x dx &\stackrel{2a}{=} \int \frac{1}{2} [\sin(8x - 5x) + \sin(8x + 5x)] dx = \frac{1}{2} \int \sin 3x dx + \frac{1}{2} \int \sin 13x dx \\ &= -\frac{1}{6} \cos 3x - \frac{1}{26} \cos 13x + C\end{aligned}$$

$$\begin{aligned}44. \int \cos \pi x \cos 4\pi x dx &\stackrel{2c}{=} \int \frac{1}{2} [\cos(\pi x - 4\pi x) + \cos(\pi x + 4\pi x)] dx = \frac{1}{2} \int \cos(-3\pi x) dx + \frac{1}{2} \int \cos(5\pi x) dx \\ &= \frac{1}{2} \int \cos 3\pi x dx + \frac{1}{2} \int \cos 5\pi x dx = \frac{1}{6\pi} \sin 3\pi x + \frac{1}{10\pi} \sin 5\pi x + C\end{aligned}$$

$$45. \int \sin 5\theta \sin \theta d\theta \stackrel{2b}{=} \int \frac{1}{2} [\cos(5\theta - \theta) - \cos(5\theta + \theta)] d\theta = \frac{1}{2} \int \cos 4\theta d\theta - \frac{1}{2} \int \cos 6\theta d\theta = \frac{1}{8} \sin 4\theta - \frac{1}{12} \sin 6\theta + C$$

$$\begin{aligned}46. \int \frac{\cos x + \sin x}{\sin 2x} dx &= \frac{1}{2} \int \frac{\cos x + \sin x}{\sin x \cos x} dx = \frac{1}{2} \int (\csc x + \sec x) dx \\ &= \frac{1}{2} \left( \ln |\csc x - \cot x| + \ln |\sec x + \tan x| \right) + C \quad [\text{by Exercise 41 and (1)}]\end{aligned}$$

$$47. \int \frac{1 - \tan^2 x}{\sec^2 x} dx = \int (\cos^2 x - \sin^2 x) dx = \int \cos 2x dx = \frac{1}{2} \sin 2x + C$$

$$\begin{aligned}48. \int \frac{dx}{\cos x - 1} &= \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1} dx = \int \frac{\cos x + 1}{\cos^2 x - 1} dx = \int \frac{\cos x + 1}{-\sin^2 x} dx \\ &= \int (-\cot x \csc x - \csc^2 x) dx = \csc x + \cot x + C\end{aligned}$$

$$49. \text{Let } u = \tan(t^2) \Rightarrow du = 2t \sec^2(t^2) dt. \text{ Then } \int t \sec^2(t^2) \tan^4(t^2) dt = \int u^4 \left( \frac{1}{2} du \right) = \frac{1}{10} u^5 + C = \frac{1}{10} \tan^5(t^2) + C.$$

$$50. \text{Let } u = \tan^7 x, dv = \sec x \tan x dx \Rightarrow du = 7 \tan^6 x \sec^2 x dx, v = \sec x. \text{ Then}$$

$$\begin{aligned}\int \tan^8 x \sec x dx &= \int \tan^7 x \cdot \sec x \tan x dx = \tan^7 x \sec x - \int 7 \tan^6 x \sec^2 x \sec x dx \\ &= \tan^7 x \sec x - 7 \int \tan^6 x (\tan^2 x + 1) \sec x dx \\ &= \tan^7 x \sec x - 7 \int \tan^8 x \sec x dx - 7 \int \tan^6 x \sec x dx.\end{aligned}$$

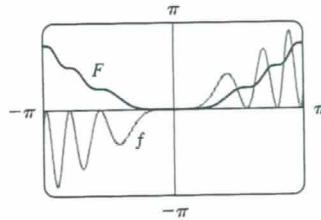
Thus,  $8 \int \tan^8 x \sec x dx = \tan^7 x \sec x - 7 \int \tan^6 x \sec x dx$  and

$$\int_0^{\pi/4} \tan^8 x \sec x dx = \frac{1}{8} [\tan^7 x \sec x]_0^{\pi/4} - \frac{7}{8} \int_0^{\pi/4} \tan^6 x \sec x dx = \frac{\sqrt{2}}{8} - \frac{7}{8} I.$$

In Exercises 51–54, let  $f(x)$  denote the integrand and  $F(x)$  its antiderivative (with  $C = 0$ ).

51. Let  $u = x^2$ , so that  $du = 2x \, dx$ . Then

$$\begin{aligned} \int x \sin^2(x^2) \, dx &= \int \sin^2 u \left(\frac{1}{2} du\right) = \frac{1}{2} \int \frac{1}{2}(1 - \cos 2u) \, du \\ &= \frac{1}{4}(u - \frac{1}{2}\sin 2u) + C = \frac{1}{4}u - \frac{1}{4}\left(\frac{1}{2} \cdot 2\sin u \cos u\right) + C \\ &= \frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C \end{aligned}$$

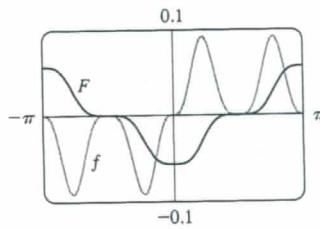


We see from the graph that this is reasonable, since  $F$  increases where  $f$  is positive and  $F$  decreases where  $f$  is negative.

Note also that  $f$  is an odd function and  $F$  is an even function.

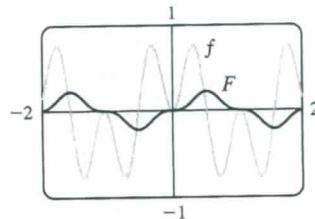
$$\begin{aligned} 52. \int \sin^3 x \cos^4 x \, dx &= \int \cos^4 x (1 - \cos^2 x) \sin x \, dx \\ &\stackrel{u = \cos x}{=} \int u^4(1 - u^2)(-du) = \int (u^6 - u^4) \, du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C \end{aligned}$$

We see from the graph that this is reasonable, since  $F$  increases where  $f$  is positive and  $F$  decreases where  $f$  is negative. Note also that  $f$  is an odd function and  $F$  is an even function.



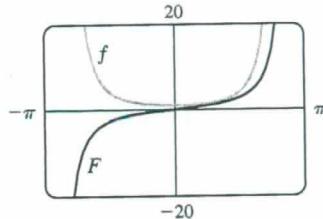
$$\begin{aligned} 53. \int \sin 3x \sin 6x \, dx &= \int \frac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)] \, dx \\ &= \frac{1}{2} \int (\cos 3x - \cos 9x) \, dx \\ &= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C \end{aligned}$$

Notice that  $f(x) = 0$  whenever  $F$  has a horizontal tangent.



$$\begin{aligned} 54. \int \sec^4 \frac{x}{2} \, dx &= \int (\tan^2 \frac{x}{2} + 1) \sec^2 \frac{x}{2} \, dx \\ &= \int (u^2 + 1) 2 \, du \quad [u = \tan \frac{x}{2}, du = \frac{1}{2} \sec^2 \frac{x}{2} \, dx] \\ &= \frac{2}{3}u^3 + 2u + C = \frac{2}{3}\tan^3 \frac{x}{2} + 2\tan \frac{x}{2} + C \end{aligned}$$

Notice that  $F$  is increasing and  $f$  is positive on the intervals on which they are defined. Also,  $F$  has no horizontal tangent and  $f$  is never zero.



$$\begin{aligned} 55. f_{\text{ave}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \cos^3 x \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x (1 - \sin^2 x) \cos x \, dx \\ &= \frac{1}{2\pi} \int_0^0 u^2(1 - u^2) \, du \quad [\text{where } u = \sin x] \\ &= 0 \end{aligned}$$

56. (a) Let  $u = \cos x$ . Then  $du = -\sin x \, dx \Rightarrow \int \sin x \cos x \, dx = \int u(-du) = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C_1$ .

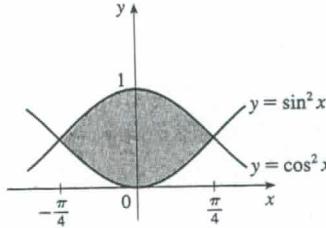
(b) Let  $u = \sin x$ . Then  $du = \cos x \, dx \Rightarrow \int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C_2$ .

(c)  $\int \sin x \cos x \, dx = \int \frac{1}{2}\sin 2x \, dx = -\frac{1}{4}\cos 2x + C_3$

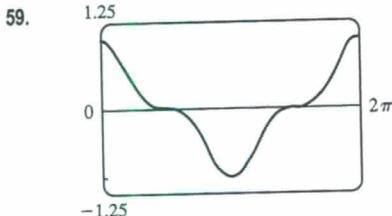
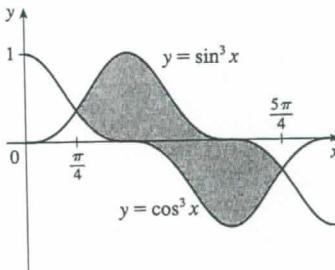
(d) Let  $u = \sin x$ ,  $dv = \cos x \, dx$ . Then  $du = \cos x \, dx$ ,  $v = \sin x$ , so  $\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx$ , by Equation 7.1.2, so  $\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C_4$ .

Using  $\cos^2 x = 1 - \sin^2 x$  and  $\cos 2x = 1 - 2\sin^2 x$ , we see that the answers differ only by a constant.

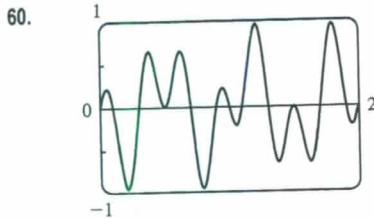
57.  $A = \int_{-\pi/4}^{\pi/4} (\cos^2 x - \sin^2 x) dx = \int_{-\pi/4}^{\pi/4} \cos 2x dx$   
 $= 2 \int_0^{\pi/4} \cos 2x dx = 2 \left[ \frac{1}{2} \sin 2x \right]_0^{\pi/4} = [\sin 2x]_0^{\pi/4}$   
 $= 1 - 0 = 1$



58.  $A = \int_{\pi/4}^{5\pi/4} (\sin^3 x - \cos^3 x) dx = \int_{\pi/4}^{5\pi/4} \sin^3 x dx - \int_{\pi/4}^{5\pi/4} \cos^3 x dx$   
 $= \int_{\pi/4}^{5\pi/4} (1 - \cos^2 x) \sin x dx - \int_{\pi/4}^{5\pi/4} (1 - \sin^2 x) \cos x dx$   
 $\stackrel{\text{c.s.}}{=} \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - u^2)(-du) - \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - u^2) du$   
 $= 2 \int_0^{\sqrt{2}/2} (1 - u^2) du + 2 \int_0^{\sqrt{2}/2} (1 - u^2) du = 4 \left[ u - \frac{1}{3} u^3 \right]_0^{\sqrt{2}/2}$   
 $= 4 \left[ \left( \frac{\sqrt{2}}{2} - \frac{1}{3} \cdot \frac{\sqrt{2}}{4} \right) - 0 \right] = 2\sqrt{2} - \frac{1}{3}\sqrt{2} = \frac{5}{3}\sqrt{2}$



It seems from the graph that  $\int_0^{2\pi} \cos^3 x dx = 0$ , since the area below the  $x$ -axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is  $[\sin x - \frac{1}{3} \sin^3 x]_0^{2\pi} = 0$ . Note that due to symmetry, the integral of any odd power of  $\sin x$  or  $\cos x$  between limits which differ by  $2n\pi$  ( $n$  any integer) is 0.



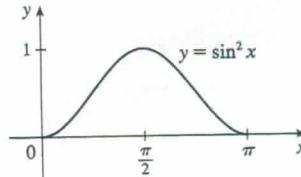
It seems from the graph that  $\int_0^2 \sin 2\pi x \cos 5\pi x dx = 0$ , since each bulge above the  $x$ -axis seems to have a corresponding depression below the  $x$ -axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned} \int_0^1 \sin 2\pi x \cos 5\pi x dx &= \frac{1}{2} \int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)] dx \\ &= \frac{1}{2} \int_0^2 [\sin(-3\pi x) + \sin 7\pi x] dx \\ &= \frac{1}{2} \left[ \frac{1}{3\pi} \cos(-3\pi x) - \frac{1}{7\pi} \cos 7\pi x \right]_0^2 \\ &= \frac{1}{2} \left[ \frac{1}{3\pi}(1 - 1) - \frac{1}{7\pi}(1 - 1) \right] = 0 \end{aligned}$$

61. Using disks,  $V = \int_{\pi/2}^{\pi} \pi \sin^2 x dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x) dx = \pi \left[ \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_{\pi/2}^{\pi} = \pi \left( \frac{\pi}{2} - 0 - \frac{\pi}{4} + 0 \right) = \frac{\pi^2}{4}$

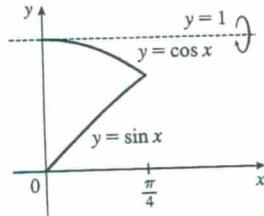
62. Using disks,

$$\begin{aligned} V &= \int_0^{\pi} \pi (\sin^2 x)^2 dx = 2\pi \int_0^{\pi/2} \left[ \frac{1}{2}(1 - \cos 2x) \right]^2 dx \\ &= \frac{\pi}{2} \int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x) dx \\ &= \frac{\pi}{2} \int_0^{\pi/2} \left[ 1 - 2\cos 2x + \frac{1}{2}(1 - \cos 4x) \right] dx \\ &= \frac{\pi}{2} \int_0^{\pi/2} \left( \frac{3}{2} - 2\cos 2x - \frac{1}{2}\cos 4x \right) dx = \frac{\pi}{2} \left[ \frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right]_0^{\pi/2} \\ &= \frac{\pi}{2} \left[ \left( \frac{3\pi}{4} - 0 + 0 \right) - (0 - 0 + 0) \right] = \frac{3}{8}\pi^2 \end{aligned}$$



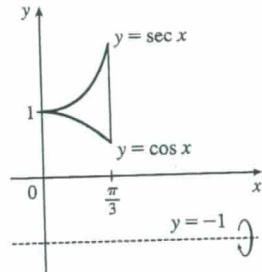
63. Using washers,

$$\begin{aligned}
 V &= \int_0^{\pi/4} \pi [(1 - \sin x)^2 - (1 - \cos x)^2] dx \\
 &= \pi \int_0^{\pi/4} [(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)] dx \\
 &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x) dx \\
 &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x) dx = \pi [2\sin x + 2\cos x - \frac{1}{2}\sin 2x]_0^{\pi/4} \\
 &= \pi [(\sqrt{2} + \sqrt{2} - \frac{1}{2}) - (0 + 2 - 0)] = \pi(2\sqrt{2} - \frac{5}{2})
 \end{aligned}$$



64. Using washers,

$$\begin{aligned}
 V &= \int_0^{\pi/3} \pi \{[\sec x - (-1)]^2 - [\cos x - (-1)]^2\} dx \\
 &= \pi \int_0^{\pi/3} [\sec^2 x + 2\sec x + 1] - [\cos^2 x + 2\cos x + 1] dx \\
 &= \pi \int_0^{\pi/3} [\sec^2 x + 2\sec x - \frac{1}{2}(1 + \cos 2x) - 2\cos x] dx \\
 &= \pi [\tan x + 2\ln |\sec x + \tan x| - \frac{1}{2}x - \frac{1}{4}\sin 2x - 2\sin x]_0^{\pi/3} \\
 &= \pi [(\sqrt{3} + 2\ln(2 + \sqrt{3}) - \frac{\pi}{6} - \frac{1}{8}\sqrt{3} - \sqrt{3}) - 0] \\
 &= 2\pi \ln(2 + \sqrt{3}) - \frac{1}{6}\pi^2 - \frac{1}{8}\pi\sqrt{3}
 \end{aligned}$$



65.  $s = f(t) = \int_0^t \sin \omega u \cos^2 \omega u du$ . Let  $y = \cos \omega u \Rightarrow dy = -\omega \sin \omega u du$ . Then

$$s = -\frac{1}{\omega} \int_1^{\cos \omega t} y^2 dy = -\frac{1}{\omega} \left[ \frac{1}{3}y^3 \right]_1^{\cos \omega t} = \frac{1}{3\omega} (1 - \cos^3 \omega t).$$

66. (a) We want to calculate the square root of the average value of  $[E(t)]^2 = [155 \sin(120\pi t)]^2 = 155^2 \sin^2(120\pi t)$ . First, we calculate the average value itself, by integrating  $[E(t)]^2$  over one cycle (between  $t = 0$  and  $t = \frac{1}{60}$ , since there are 60 cycles per second) and dividing by  $(\frac{1}{60} - 0)$ :

$$\begin{aligned}
 [E(t)]_{\text{ave}}^2 &= \frac{1}{1/60} \int_0^{1/60} [155^2 \sin^2(120\pi t)] dt = 60 \cdot 155^2 \int_0^{1/60} \frac{1}{2}[1 - \cos(240\pi t)] dt \\
 &= 60 \cdot 155^2 \left( \frac{1}{2} \right) \left[ t - \frac{1}{240\pi} \sin(240\pi t) \right]_0^{1/60} = 60 \cdot 155^2 \left( \frac{1}{2} \right) \left[ \left( \frac{1}{60} - 0 \right) - (0 - 0) \right] = \frac{155^2}{2}
 \end{aligned}$$

The RMS value is just the square root of this quantity, which is  $\frac{155}{\sqrt{2}} \approx 110$  V.

$$\begin{aligned}
 (b) 220 &= \sqrt{[E(t)]_{\text{ave}}^2} \Rightarrow \\
 220^2 &= [E(t)]_{\text{ave}}^2 = \frac{1}{1/60} \int_0^{1/60} A^2 \sin^2(120\pi t) dt = 60A^2 \int_0^{1/60} \frac{1}{2}[1 - \cos(240\pi t)] dt \\
 &= 30A^2 \left[ t - \frac{1}{240\pi} \sin(240\pi t) \right]_0^{1/60} = 30A^2 \left[ \left( \frac{1}{60} - 0 \right) - (0 - 0) \right] = \frac{1}{2}A^2
 \end{aligned}$$

Thus,  $220^2 = \frac{1}{2}A^2 \Rightarrow A = 220\sqrt{2} \approx 311$  V.

67. Just note that the integrand is odd [ $f(-x) = -f(x)$ ].

Or: If  $m \neq n$ , calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] dx = \frac{1}{2} \left[ -\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0$$

If  $m = n$ , then the first term in each set of brackets is zero.

68.  $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] dx.$

If  $m \neq n$ , this is equal to  $\frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0$ .

If  $m = n$ , we get  $\int_{-\pi}^{\pi} \frac{1}{2} [1 - \cos(m+n)x] dx = \left[ \frac{1}{2}x \right]_{-\pi}^{\pi} - \left[ \frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi - 0 = \pi$ .

69.  $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] dx.$

If  $m \neq n$ , this is equal to  $\frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0$ .

If  $m = n$ , we get  $\int_{-\pi}^{\pi} \frac{1}{2} [1 + \cos(m+n)x] dx = \left[ \frac{1}{2}x \right]_{-\pi}^{\pi} + \left[ \frac{\sin(m+n)x}{2(m+n)} \right]_{-\pi}^{\pi} = \pi + 0 = \pi$ .

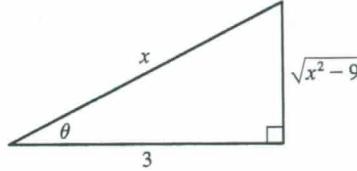
70.  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \left( \sum_{n=1}^m a_n \sin nx \right) \sin mx \right] dx = \sum_{n=1}^m \frac{a_n}{\pi} \int_{-\pi}^{\pi} \sin mx \sin nx dx.$  By Exercise 68, every term is zero except the  $m$ th one, and that term is  $\frac{a_m}{\pi} \cdot \pi = a_m$ .

### 7.3 Trigonometric Substitution

1. Let  $x = 3 \sec \theta$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ . Then

$dx = 3 \sec \theta \tan \theta d\theta$  and

$$\begin{aligned} \sqrt{x^2 - 9} &= \sqrt{9 \sec^2 \theta - 9} = \sqrt{9(\sec^2 \theta - 1)} = \sqrt{9 \tan^2 \theta} \\ &= 3 |\tan \theta| = 3 \tan \theta \text{ for the relevant values of } \theta. \end{aligned}$$

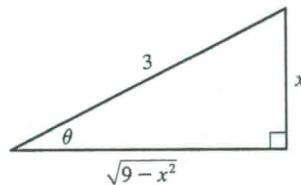


$$\int \frac{1}{x^2 \sqrt{x^2 - 9}} dx = \int \frac{1}{9 \sec^2 \theta \cdot 3 \tan \theta} 3 \sec \theta \tan \theta d\theta = \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + C = \frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C$$

Note that  $-\sec(\theta + \pi) = \sec \theta$ , so the figure is sufficient for the case  $\pi \leq \theta < \frac{3\pi}{2}$ .

2. Let  $x = 3 \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $dx = 3 \cos \theta d\theta$  and

$$\begin{aligned} \sqrt{9 - x^2} &= \sqrt{9 - 9 \sin^2 \theta} = \sqrt{9(1 - \sin^2 \theta)} = \sqrt{9 \cos^2 \theta} \\ &= 3 |\cos \theta| = 3 \cos \theta \text{ for the relevant values of } \theta. \end{aligned}$$



$$\begin{aligned} \int x^3 \sqrt{9 - x^2} dx &= \int 3^3 \sin^3 \theta \cdot 3 \cos \theta \cdot 3 \cos \theta d\theta = 3^5 \int \sin^3 \theta \cos^2 \theta d\theta = 3^5 \int \sin^2 \theta \cos^2 \theta \sin \theta d\theta \\ &= 3^5 \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta = 3^5 \int (1 - u^2) u^2 (-du) \quad [u = \cos \theta, du = -\sin \theta d\theta] \\ &= 3^5 \int (u^4 - u^2) du = 3^5 \left( \frac{1}{5} u^5 - \frac{1}{3} u^3 \right) + C = 3^5 \left( \frac{1}{5} \cos^5 \theta - \frac{1}{3} \cos^3 \theta \right) + C \\ &= 3^5 \left[ \frac{1}{5} \frac{(9 - x^2)^{5/2}}{3^5} - \frac{1}{3} \frac{(9 - x^2)^{3/2}}{3^3} \right] + C \\ &= \frac{1}{5} (9 - x^2)^{5/2} - 3(9 - x^2)^{3/2} + C \quad \text{or} \quad -\frac{1}{5} (x^2 + 6)(9 - x^2)^{3/2} + C \end{aligned}$$