EXAMPLE 6 Prove the reduction formula

■ Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n - 1$ and $n - 2$.

$$\boxed{7} \qquad \int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let $\quad u = \sin^{n-1}x \qquad dv = \sin x\,dx$

Then $\quad du = (n-1)\sin^{n-2}x\cos x\,dx \qquad v = -\cos x$

so integration by parts gives

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

□

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

7.1 EXERCISES

1–2 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int x^2 \ln x\,dx;\quad u = \ln x,\ dv = x^2\,dx$

2. $\int \theta\cos\theta\,d\theta;\quad u = \theta,\ dv = \cos\theta\,d\theta$

3–32 Evaluate the integral.

3. $\int x\cos 5x\,dx$

4. $\int xe^{-x}\,dx$

5. $\int re^{r/2}\,dr$

6. $\int t\sin 2t\,dt$

7. $\int x^2\sin\pi x\,dx$

8. $\int x^2\cos mx\,dx$

9. $\int \ln(2x+1)\,dx$

10. $\int \sin^{-1}x\,dx$

11. $\int \arctan 4t\,dt$

12. $\int p^5\ln p\,dp$

13. $\int t\sec^2 2t\,dt$

14. $\int s2^s\,ds$

[15.] $\int (\ln x)^2\,dx$

16. $\int t\sinh mt\,dt$

[17.] $\int e^{2\theta}\sin 3\theta\,d\theta$

18. $\int e^{-\theta}\cos 2\theta\,d\theta$

19. $\int_0^\pi t\sin 3t\,dt$

[20.] $\int_0^1 (x^2+1)e^{-x}\,dx$

21. $\int_0^1 t\cosh t\,dt$

22. $\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$

23. $\int_1^2 \frac{\ln x}{x^2}\,dx$

24. $\int_0^\pi x^3\cos x\,dx$

25. $\int_0^1 \frac{y}{e^{2y}}\,dy$

26. $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

27. $\int_0^{1/2} \cos^{-1}x\,dx$

28. $\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$

29. $\int \cos x \ln(\sin x)\,dx$

30. $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

31. $\int_1^2 x^4(\ln x)^2\,dx$

32. $\int_0^t e^s \sin(t-s)\,ds$

33–38 First make a substitution and then use integration by parts to evaluate the integral.

33. $\int \cos\sqrt{x}\,dx$

34. $\int t^3 e^{-t^2}\,dt$

35. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta$

36. $\int_0^{\pi} e^{\cos t}\sin 2t\,dt$

37. $\int x\ln(1+x)\,dx$

38. $\int \sin(\ln x)\,dx$

39–42 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

39. $\int (2x+3)e^x\,dx$

40. $\int x^{3/2}\ln x\,dx$

41. $\int x^3\sqrt{1+x^2}\,dx$

42. $\int x^2 \sin 2x\,dx$

43. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

44. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

45. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geq 2$ is an integer.

(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}$$

46. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\,\frac{\pi}{2}$$

47–50 Use integration by parts to prove the reduction formula.

47. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

48. $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

49. $\tan^n x\,dx = \dfrac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$

50. $\int \sec^n x\,dx = \dfrac{\tan x\sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n \neq 1)$

51. Use Exercise 47 to find $\int (\ln x)^3\,dx$.

52. Use Exercise 48 to find $\int x^4 e^x\,dx$.

53–54 Find the area of the region bounded by the given curves.

53. $y = xe^{-0.4x}$, $y = 0$, $x = 5$

54. $y = 5\ln x$, $y = x\ln x$

55–56 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

55. $y = x\sin x$, $y = (x-2)^2$

56. $y = \arctan 3x$, $y = \frac{1}{2}x$

57–60 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

57. $y = \cos(\pi x/2)$, $y = 0$, $0 \leq x \leq 1$; about the y-axis

58. $y = e^x$, $y = e^{-x}$, $x = 1$; about the y-axis

59. $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$

60. $y = e^x$, $x = 0$, $y = \pi$; about the x-axis

61. Find the average value of $f(x) = x^2 \ln x$ on the interval $[1, 3]$.

62. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

63. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

64. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

65. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

66. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]

(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

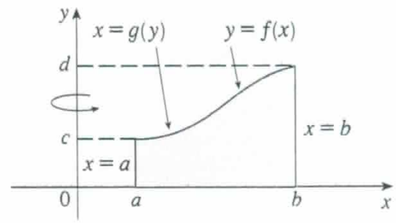
(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

67. We arrived at Formula 6.3.2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$



68. Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.

(b) Use Exercise 46 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.

(d) Use part (c) and Exercises 45 and 46 to show that

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product.*

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.