

In this book, Equation 2 is the Integration by Parts Formula,
i.e.

$$\text{Equation 2} \quad \text{is} \quad \int u \, dv = uv - \int v \, du.$$

7 □ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = \ln x, dv = x^2 dx \Rightarrow du = \frac{1}{x} dx, v = \frac{1}{3}x^3$. Then by Equation 2,

$$\begin{aligned} \int x^2 \ln x \, dx &= (\ln x)(\frac{1}{3}x^3) - \int (\frac{1}{3}x^3)(\frac{1}{x}) \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}(\frac{1}{3}x^3) + C \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \quad [\text{or } \frac{1}{3}x^3(\ln x - \frac{1}{3}) + C] \end{aligned}$$

2. Let $u = \theta, dv = \cos \theta d\theta \Rightarrow du = d\theta, v = \sin \theta$. Then by Equation 2,

$$\int \theta \cos \theta \, d\theta = \theta \sin \theta - \int \sin \theta \, d\theta = \theta \sin \theta + \cos \theta + C.$$

3. Let $u = x, dv = \cos 5x \, dx \Rightarrow du = dx, v = \frac{1}{5} \sin 5x$. Then by Equation 2,

$$\int x \cos 5x \, dx = \frac{1}{5}x \sin 5x - \int \frac{1}{5} \sin 5x \, dx = \frac{1}{5}x \sin 5x + \frac{1}{25} \cos 5x + C.$$

4. Let $u = x, dv = e^{-x} \, dx \Rightarrow du = dx, v = -e^{-x}$. Then $\int xe^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C$.

5. Let $u = r, dv = e^{r/2} \, dr \Rightarrow du = dr, v = 2e^{r/2}$. Then $\int re^{r/2} \, dr = 2re^{r/2} - \int 2e^{r/2} \, dr = 2re^{r/2} - 4e^{r/2} + C$.

6. Let $u = t, dv = \sin 2t \, dt \Rightarrow du = dt, v = -\frac{1}{2} \cos 2t$. Then

$$\int t \sin 2t \, dt = -\frac{1}{2}t \cos 2t + \frac{1}{2} \int \cos 2t \, dt = -\frac{1}{2}t \cos 2t + \frac{1}{4} \sin 2t + C.$$

7. Let $u = x^2, dv = \sin \pi x \, dx \Rightarrow du = 2x \, dx$ and $v = -\frac{1}{\pi} \cos \pi x$. Then

$$I = \int x^2 \sin \pi x \, dx = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi} \int x \cos \pi x \, dx \quad (*). \text{ Next let } U = x, dV = \cos \pi x \, dx \Rightarrow dU = dx.$$

$$V = \frac{1}{\pi} \sin \pi x, \text{ so } \int x \cos \pi x \, dx = \frac{1}{\pi}x \sin \pi x - \frac{1}{\pi} \int \sin \pi x \, dx = \frac{1}{\pi}x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1.$$

Substituting for $\int x \cos \pi x \, dx$ in (*), we get

$$I = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi}(\frac{1}{\pi}x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1) = -\frac{1}{\pi}x^2 \cos \pi x + \frac{2}{\pi^2}x \sin \pi x + \frac{2}{\pi^3} \cos \pi x + C, \text{ where } C = \frac{2}{\pi}C_1.$$

8. Let $u = x^2$, $dv = \cos mx dx \Rightarrow du = 2x dx$, $v = \frac{1}{m} \sin mx$. Then

$$I = \int x^2 \cos mx dx = \frac{1}{m} x^2 \sin mx - \frac{2}{m} \int x \sin mx dx \quad (\star). \text{ Next let } U = x, dV = \sin mx dx \Rightarrow dU = dx,$$

$$V = -\frac{1}{m} \cos mx, \text{ so } \int x \sin mx dx = -\frac{1}{m} x \cos mx + \frac{1}{m} \int \cos mx dx = -\frac{1}{m} x \cos mx + \frac{1}{m^2} \sin mx + C_1.$$

Substituting for $\int x \sin mx dx$ in (\star) , we get

$$I = \frac{1}{m} x^2 \sin mx - \frac{2}{m} \left(-\frac{1}{m} x \cos mx + \frac{1}{m^2} \sin mx + C_1 \right) = \frac{1}{m} x^2 \sin mx + \frac{2}{m^2} x \cos mx - \frac{2}{m^3} \sin mx + C,$$

where $C = -\frac{2}{m} C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \frac{2}{2x+1} dx$, $v = x$. Then

$$\begin{aligned} \int \ln(2x+1) dx &= x \ln(2x+1) - \int \frac{2x}{2x+1} dx = x \ln(2x+1) - \int \frac{(2x+1)-1}{2x+1} dx \\ &= x \ln(2x+1) - \int \left(1 - \frac{1}{2x+1} \right) dx = x \ln(2x+1) - x + \frac{1}{2} \ln(2x+1) + C \\ &= \frac{1}{2}(2x+1) \ln(2x+1) - x + C \end{aligned}$$

10. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$. Setting

$$t = 1 - x^2, \text{ we get } dt = -2x dx, \text{ so } -\int \frac{x dx}{\sqrt{1-x^2}} = -\int t^{-1/2} (-\frac{1}{2} dt) = \frac{1}{2}(2t^{1/2}) + C = t^{1/2} + C = \sqrt{1-x^2} + C.$$

Hence, $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$.

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \frac{4}{1+(4t)^2} dt = \frac{4}{1+16t^2} dt$, $v = t$. Then

$$\int \arctan 4t dt = t \arctan 4t - \int \frac{4t}{1+16t^2} dt = t \arctan 4t - \frac{1}{8} \int \frac{32t}{1+16t^2} dt = t \arctan 4t - \frac{1}{8} \ln(1+16t^2) + C.$$

12. Let $u = \ln p$, $dv = p^5 dp \Rightarrow du = \frac{1}{p} dp$, $v = \frac{1}{6} p^6$. Then $\int p^5 \ln p dp = \frac{1}{6} p^6 \ln p - \frac{1}{6} \int p^5 dp = \frac{1}{6} p^6 \ln p - \frac{1}{36} p^6 + C$.

13. Let $u = t$, $dv = \sec^2 2t dt \Rightarrow du = dt$, $v = \frac{1}{2} \tan 2t$. Then

$$\int t \sec^2 2t dt = \frac{1}{2} t \tan 2t - \frac{1}{2} \int \tan 2t dt = \frac{1}{2} t \tan 2t - \frac{1}{4} \ln |\sec 2t| + C.$$

14. Let $u = s$, $dv = 2^s ds \Rightarrow du = ds$, $v = \frac{1}{\ln 2} 2^s$. Then

$$\int s 2^s ds = \frac{1}{\ln 2} s 2^s - \frac{1}{\ln 2} \int 2^s ds = \frac{1}{\ln 2} s 2^s - \frac{1}{(\ln 2)^2} 2^s + C \quad \left[\text{or } \frac{2^s}{(\ln 2)^2} (s \ln 2 - 1) + C \right].$$

15. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2 \ln x \cdot \frac{1}{x} dx$, $v = x$. Then by Equation 2,

$$I = \int (\ln x)^2 dx = x(\ln x)^2 - 2 \int x \ln x \cdot \frac{1}{x} dx = x(\ln x)^2 - 2 \int \ln x dx. \text{ Next let } U = \ln x, dV = dx \Rightarrow$$

$dU = 1/x dx$, $V = x$ to get $\int \ln x dx = x \ln x - \int x \cdot (1/x) dx = x \ln x - \int dx = x \ln x - x + C_1$. Thus,

$$I = x(\ln x)^2 - 2(x \ln x - x + C_1) = x(\ln x)^2 - 2x \ln x + 2x + C, \text{ where } C = -2C_1.$$

16. Let $u = t$, $dv = \sinh mt dt \Rightarrow du = dt$, $v = \frac{1}{m} \cosh mt$. Then

$$\int t \sinh mt dt = \frac{1}{m} t \cosh mt - \int \frac{1}{m} \cosh mt dt = \frac{1}{m} t \cosh mt - \frac{1}{m^2} \sinh mt + C \quad [m \neq 0].$$

17. First let $u = \sin 3\theta$, $dv = e^{2\theta} d\theta \Rightarrow du = 3 \cos 3\theta d\theta$, $v = \frac{1}{2} e^{2\theta}$. Then

$$I = \int e^{2\theta} \sin 3\theta d\theta = \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{2} \int e^{2\theta} \cos 3\theta d\theta. \text{ Next let } U = \cos 3\theta, dV = e^{2\theta} d\theta \Rightarrow dU = -3 \sin 3\theta d\theta,$$

$V = \frac{1}{2} e^{2\theta}$ to get $\int e^{2\theta} \cos 3\theta d\theta = \frac{1}{2} e^{2\theta} \cos 3\theta + \frac{3}{2} \int e^{2\theta} \sin 3\theta d\theta$. Substituting in the previous formula gives

$$I = \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta - \frac{9}{4} \int e^{2\theta} \sin 3\theta d\theta = \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta - \frac{9}{4} I \Rightarrow$$

$$\frac{13}{4} I = \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta + C_1. \text{ Hence, } I = \frac{1}{13} e^{2\theta} (2 \sin 3\theta - 3 \cos 3\theta) + C, \text{ where } C = \frac{4}{13} C_1.$$

18. First let $u = e^{-\theta}$, $dv = \cos 2\theta d\theta \Rightarrow du = -e^{-\theta} d\theta$, $v = \frac{1}{2} \sin 2\theta$. Then

$$I = \int e^{-\theta} \cos 2\theta d\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \int \frac{1}{2} \sin 2\theta (-e^{-\theta} d\theta) = \frac{1}{2} e^{-\theta} \sin 2\theta + \frac{1}{2} \int e^{-\theta} \sin 2\theta d\theta.$$

Next let $U = e^{-\theta}$, $dV = \sin 2\theta d\theta \Rightarrow dU = -e^{-\theta} d\theta$, $V = -\frac{1}{2} \cos 2\theta$, so

$$\int e^{-\theta} \sin 2\theta d\theta = -\frac{1}{2} e^{-\theta} \cos 2\theta - \int (-\frac{1}{2}) \cos 2\theta (-e^{-\theta} d\theta) = -\frac{1}{2} e^{-\theta} \cos 2\theta - \frac{1}{2} \int e^{-\theta} \cos 2\theta d\theta.$$

$$\text{So } I = \frac{1}{2} e^{-\theta} \sin 2\theta + \frac{1}{2} [(-\frac{1}{2} e^{-\theta} \cos 2\theta) - \frac{1}{2} I] = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta - \frac{1}{4} I \Rightarrow$$

$$\frac{5}{4} I = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta + C_1 \Rightarrow I = \frac{4}{5} (\frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta + C_1) = \frac{2}{5} e^{-\theta} \sin 2\theta - \frac{1}{5} e^{-\theta} \cos 2\theta + C.$$

19. Let $u = t$, $dv = \sin 3t dt \Rightarrow du = dt$, $v = -\frac{1}{3} \cos 3t$. Then

$$\int_0^\pi t \sin 3t dt = [-\frac{1}{3} t \cos 3t]_0^\pi + \frac{1}{3} \int_0^\pi \cos 3t dt = (\frac{1}{3}\pi - 0) + \frac{1}{9} [\sin 3t]_0^\pi = \frac{\pi}{3}.$$

20. First let $u = x^2 + 1$, $dv = e^{-x} dx \Rightarrow du = 2x dx$, $v = -e^{-x}$. By (6),

$$\int_0^1 (x^2 + 1) e^{-x} dx = [-(x^2 + 1) e^{-x}]_0^1 + \int_0^1 2x e^{-x} dx = -2e^{-1} + 1 + 2 \int_0^1 x e^{-x} dx.$$

Next let $U = x$, $dV = e^{-x} dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,

$$\int_0^1 x e^{-x} dx = [-x e^{-x}]_0^1 + \int_0^1 e^{-x} dx = -e^{-1} + [-e^{-x}]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1. \text{ So}$$

$$\int_0^1 (x^2 + 1) e^{-x} dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3.$$

21. Let $u = t$, $dv = \cosh t dt \Rightarrow du = dt$, $v = \sinh t$. Then

$$\begin{aligned} \int_0^1 t \cosh t dt &= [t \sinh t]_0^1 - \int_0^1 \sinh t dt = (\sinh 1 - \sinh 0) - [\cosh t]_0^1 = \sinh 1 - (\cosh 1 - \cosh 0) \\ &= \sinh 1 - \cosh 1 + 1. \end{aligned}$$

We can use the definitions of sinh and cosh to write the answer in terms of e :

$$\sinh 1 - \cosh 1 + 1 = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^1 + e^{-1}) + 1 = -e^{-1} + 1 = 1 - 1/e.$$

22. Let $u = \ln y$, $dv = \frac{1}{\sqrt{y}} dy = y^{-1/2} dy \Rightarrow du = \frac{1}{y} dy$, $v = 2y^{1/2}$. Then

$$\begin{aligned} \int_4^9 \frac{\ln y}{\sqrt{y}} dy &= \left[2\sqrt{y} \ln y \right]_4^9 - \int_4^9 2y^{-1/2} dy = (6 \ln 9 - 4 \ln 4) - \left[4\sqrt{y} \right]_4^9 = 6 \ln 9 - 4 \ln 4 - (12 - 8) \\ &= 6 \ln 9 - 4 \ln 4 - 4 \end{aligned}$$

23. Let $u = \ln x$, $dv = x^{-2} dx \Rightarrow du = \frac{1}{x} dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_1^2 + \int_1^2 x^{-2} dx = -\frac{1}{2} \ln 2 + \ln 1 + \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} \ln 2 + 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \ln 2.$$

24. First let $u = x^3$, $dv = \cos x dx \Rightarrow du = 3x^2 dx$, $v = \sin x$. Then $I_1 = \int x^3 \cos x dx = x^3 \sin x - 3 \int x^2 \sin x dx$. Next let $u_1 = x^2$, $dv_1 = \sin x dx \Rightarrow du_1 = 2x dx$, $v_1 = -\cos x$. Then $I_2 = \int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx$.

Finally, let $u_2 = x$, $dv_2 = \cos x dx \Rightarrow du_2 = dx$, $v_2 = \sin x$. Then

$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$. Substituting in the expression for I_2 , we get

$I_2 = -x^2 \cos x + 2(x \sin x + \cos x + C) = -x^2 \cos x + 2x \sin x + 2 \cos x + 2C$. Substituting the last expression for I_2 into I_1 gives $I_1 = x^3 \sin x - 3(-x^2 \cos x + 2x \sin x + 2 \cos x + 2C) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x - 6C$. Thus,

$$\begin{aligned} \int_0^\pi x^3 \cos x dx &= [x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x - 6C]_0^\pi \\ &= (0 - 3\pi^2 - 0 + 6 - 6C) - (0 + 0 - 0 - 6 - 6C) = 12 - 3\pi^2 \end{aligned}$$

25. Let $u = y$, $dv = \frac{dy}{e^{2y}} = e^{-2y} dy \Rightarrow du = dy$, $v = -\frac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}} dy = \left[-\frac{1}{2}ye^{-2y} \right]_0^1 + \frac{1}{2} \int_0^1 e^{-2y} dy = \left(-\frac{1}{2}e^{-2} + 0 \right) - \frac{1}{4} \left[e^{-2y} \right]_0^1 = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4}e^{-2}.$$

26. Let $u = \arctan(1/x)$, $dv = dx \Rightarrow du = \frac{1}{1+(1/x)^2} \cdot \frac{-1}{x^2} dx = \frac{-dx}{x^2+1}$, $v = x$. Then

$$\begin{aligned} \int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx &= \left[x \arctan\left(\frac{1}{x}\right) \right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x dx}{x^2+1} = \sqrt{3} \frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2} \left[\ln(x^2+1) \right]_1^{\sqrt{3}} \\ &= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2} \ln \frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2} \ln 2 \end{aligned}$$

27. Let $u = \cos^{-1} x$, $dv = dx \Rightarrow du = -\frac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2} \cos^{-1} x dx = [x \cos^{-1} x]_0^{1/2} + \int_0^{1/2} \frac{x dx}{\sqrt{1-x^2}} = \frac{1}{2} \cdot \frac{\pi}{3} + \int_1^{3/4} t^{-1/2} \left[-\frac{1}{2} dt \right], \text{ where } t = 1 - x^2 \Rightarrow dt = -2x dx. \text{ Thus, } I = \frac{\pi}{6} + \frac{1}{2} \int_{3/4}^1 t^{-1/2} dt = \frac{\pi}{6} + [\sqrt{t}]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}(\pi + 6 - 3\sqrt{3}).$$

28. Let $u = (\ln x)^2$, $dv = x^{-3} dx \Rightarrow du = \frac{2 \ln x}{x} dx$, $v = -\frac{1}{2}x^{-2}$. Then

$$I = \int_1^2 \frac{(\ln x)^2}{x^3} dx = \left[-\frac{(\ln x)^2}{2x^2} \right]_1^2 + \int_1^2 \frac{\ln x}{x^3} dx. \text{ Now let } U = \ln x, dV = x^{-3} dx \Rightarrow dU = \frac{1}{x} dx, V = -\frac{1}{2}x^{-2}.$$

Then

$$\int_1^2 \frac{\ln x}{x^3} dx = \left[-\frac{\ln x}{2x^2} \right]_1^2 + \frac{1}{2} \int_1^2 x^{-3} dx = -\frac{1}{8} \ln 2 + 0 + \frac{1}{2} \left[-\frac{1}{2x^2} \right]_1^2 = -\frac{1}{8} \ln 2 + \frac{1}{2} \left(-\frac{1}{8} + \frac{1}{2} \right) = \frac{3}{16} - \frac{1}{8} \ln 2.$$

Thus $I = \left(-\frac{1}{8} \ln 2 + 0 \right) + \left(\frac{3}{16} - \frac{1}{8} \ln 2 \right) = -\frac{1}{8} \ln 2 + \frac{3}{16}$.

29. Let $u = \ln(\sin x)$, $dv = \cos x dx \Rightarrow du = \frac{\cos x}{\sin x} dx$, $v = \sin x$. Then

$$I = \int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \int \cos x dx = \sin x \ln(\sin x) - \sin x + C.$$

Another method: Substitute $t = \sin x$, so $dt = \cos x dx$. Then $I = \int \ln t dt = t \ln t - t + C$ (see Example 2) and so

$$I = \sin x (\ln \sin x - 1) + C.$$

30. Let $u = r^2$, $dv = \frac{r}{\sqrt{4+r^2}} dr \Rightarrow du = 2r dr$, $v = \sqrt{4+r^2}$. By (6),

$$\begin{aligned} \int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr &= \left[r^2 \sqrt{4+r^2} \right]_0^1 - 2 \int_0^1 r \sqrt{4+r^2} dr = \sqrt{5} - \frac{2}{3} \left[(4+r^2)^{3/2} \right]_0^1 \\ &= \sqrt{5} - \frac{2}{3}(5)^{3/2} + \frac{2}{3}(8) = \sqrt{5} \left(1 - \frac{10}{3} \right) + \frac{16}{3} = \frac{16}{3} - \frac{7}{3}\sqrt{5} \end{aligned}$$

31. Let $u = (\ln x)^2$, $dv = x^4 dx \Rightarrow du = 2 \frac{\ln x}{x} dx$, $v = \frac{x^5}{5}$. By (6),

$$\int_1^2 x^4 (\ln x)^2 dx = \left[\frac{x^5}{5} (\ln x)^2 \right]_1^2 - 2 \int_1^2 \frac{x^4}{5} \ln x dx = \frac{32}{5} (\ln 2)^2 - 0 - 2 \int_1^2 \frac{x^4}{5} \ln x dx.$$

$$\text{Let } U = \ln x, dV = \frac{x^4}{5} dx \Rightarrow dU = \frac{1}{x} dx, V = \frac{x^5}{25}.$$

$$\text{Then } \int_1^2 \frac{x^4}{5} \ln x dx = \left[\frac{x^5}{25} \ln x \right]_1^2 - \int_1^2 \frac{x^4}{25} dx = \frac{32}{25} \ln 2 - 0 - \left[\frac{x^5}{125} \right]_1^2 = \frac{32}{25} \ln 2 - \left(\frac{32}{125} - \frac{1}{125} \right).$$

$$\text{So } \int_1^2 x^4 (\ln x)^2 dx = \frac{32}{5} (\ln 2)^2 - 2 \left(\frac{32}{25} \ln 2 - \frac{1}{125} \right) = \frac{32}{5} (\ln 2)^2 - \frac{64}{25} \ln 2 + \frac{62}{125}.$$

32. Let $u = \sin(t-s)$, $dv = e^s ds \Rightarrow du = -\cos(t-s) ds$, $v = e^s$. Then

$$I = \int_0^t e^s \sin(t-s) ds = \left[e^s \sin(t-s) \right]_0^t + \int_0^t e^s \cos(t-s) ds = e^t \sin 0 - e^0 \sin t + I_1. \text{ For } I_1, \text{ let } U = \cos(t-s),$$

$$dV = e^s ds \Rightarrow dU = \sin(t-s) ds, V = e^s. \text{ So } I_1 = \left[e^s \cos(t-s) \right]_0^t - \int_0^t e^s \sin(t-s) ds = e^t \cos 0 - e^0 \cos t - I.$$

$$\text{Thus, } I = -\sin t + e^t - \cos t - I \Rightarrow 2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}(e^t - \cos t - \sin t).$$

33. Let $y = \sqrt{x}$, so that $dy = \frac{1}{2}x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx = \frac{1}{2y} dx$. Thus, $\int \cos \sqrt{x} dx = \int \cos y (2y dy) = 2 \int y \cos y dy$. Now

$$\text{use parts with } u = y, dv = \cos y dy, du = dy, v = \sin y \text{ to get } \int y \cos y dy = y \sin y - \int \sin y dy = y \sin y + \cos y + C_1,$$

$$\text{so } \int \cos \sqrt{x} dx = 2y \sin y + 2 \cos y + C = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.$$

34. Let $x = -t^2$, so that $dx = -2t dt$. Thus, $\int t^3 e^{-t^2} dt = \int (-t^2) e^{-t^2} \left(\frac{1}{2}(-2t) dt \right) = \frac{1}{2} \int x e^x dx$. Now use parts with

$u = x, dv = e^x dx, du = dx, v = e^x$ to get

$$u = x, dv = e^x dx, du = dx, v = e^x \Rightarrow \frac{1}{2} \int x e^x dx = \frac{1}{2} (x e^x - \int e^x dx) = \frac{1}{2} x e^x - \frac{1}{2} e^x + C = -\frac{1}{2}(1-x)e^x + C = -\frac{1}{2}(1+t^2)e^{-t^2} + C.$$

$$\frac{1}{2} \int x e^x dx = \frac{1}{2} (x e^x - \int e^x dx) = \frac{1}{2} x e^x - \frac{1}{2} e^x + C = -\frac{1}{2}(1-t^2)e^{-t^2} + C.$$

35. Let $x = \theta^2$, so that $dx = 2\theta d\theta$. Thus, $\int \sqrt{\pi} \theta^3 \cos(\theta^2) d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos(\theta^2) \cdot \frac{1}{2}(2\theta d\theta) = \frac{1}{2} \int_{\pi/2}^{\pi} x \cos x dx$. Now use

parts with $u = x, dv = \cos x dx, du = dx, v = \sin x$ to get

$$\frac{1}{2} \int_{\pi/2}^{\pi} x \cos x dx = \frac{1}{2} \left([x \sin x]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x dx \right) = \frac{1}{2} [x \sin x + \cos x]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}(\pi \sin \pi + \cos \pi) - \frac{1}{2}\left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) = \frac{1}{2}(\pi \cdot 0 - 1) - \frac{1}{2}\left(\frac{\pi}{2} \cdot 1 + 0\right) = -\frac{1}{2} - \frac{\pi}{4}$$

36. Let $x = \cos t$, so that $dx = -\sin t dt$. Thus,

$$\int_0^\pi e^{\cos t} \sin 2t dt = \int_0^\pi e^{\cos t} (2 \sin t \cos t) dt = \int_{-1}^{-1} e^x \cdot 2x (-dx) = 2 \int_{-1}^1 xe^x dx. \text{ Now use parts with } u = x,$$

$dv = e^x dx, du = dx, v = e^x$ to get

$$2 \int_{-1}^1 xe^x dx = 2 \left([xe^x]_{-1}^1 - \int_{-1}^1 e^x dx \right) = 2 \left(e^1 + e^{-1} - [e^x]_{-1}^1 \right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e.$$

37. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x \ln(1+x) dx = \int (y-1) \ln y dy$. Now use parts with $u = \ln y, dv = (y-1) dy$,

$du = \frac{1}{y} dy, v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned} \int (y-1) \ln y dy &= \left(\frac{1}{2}y^2 - y \right) \ln y - \int \left(\frac{1}{2}y - 1 \right) dy = \frac{1}{2}y(y-2) \ln y - \frac{1}{4}y^2 + y + C \\ &= \frac{1}{2}(1+x)(x-1) \ln(1+x) - \frac{1}{4}(1+x)^2 + 1 + x + C, \end{aligned}$$

which can be written as $\frac{1}{2}(x^2 - 1) \ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

38. Let $y = \ln x$, so that $dy = \frac{1}{x} dx \Rightarrow dx = x dy = e^y dy$. Thus,

$$\int \sin(\ln x) dx = \int \sin y e^y dy = \frac{1}{2}e^y (\sin y - \cos y) + C \quad [\text{by Example 4}] = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C.$$

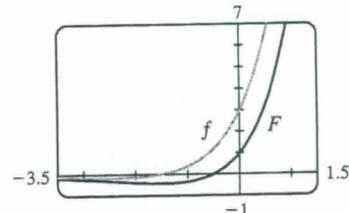
In Exercises 39–42, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

39. Let $u = 2x + 3, dv = e^x dx \Rightarrow du = 2 dx, v = e^x$. Then

$$\begin{aligned} \int (2x+3)e^x dx &= (2x+3)e^x - 2 \int e^x dx = (2x+3)e^x - 2e^x + C \\ &= (2x+1)e^x + C \end{aligned}$$

We see from the graph that this is reasonable, since F has a minimum where

f changes from negative to positive.

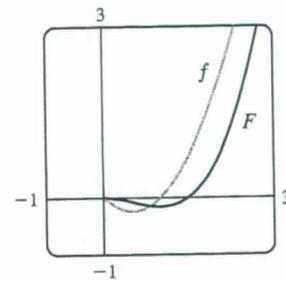


40. Let $u = \ln x, dv = x^{3/2} dx \Rightarrow du = \frac{1}{x} dx, v = \frac{2}{5}x^{5/2}$. Then

$$\begin{aligned} \int x^{3/2} \ln x dx &= \frac{2}{5}x^{5/2} \ln x - \frac{2}{5} \int x^{3/2} dx = \frac{2}{5}x^{5/2} \ln x - \left(\frac{2}{5} \right)^2 x^{5/2} + C \\ &= \frac{2}{5}x^{5/2} \ln x - \frac{4}{25}x^{5/2} + C \end{aligned}$$

We see from the graph that this is reasonable, since F has a minimum where

f changes from negative to positive.

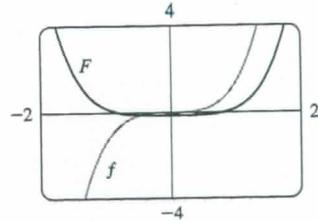


41. Let $u = \frac{1}{2}x^2, dv = 2x \sqrt{1+x^2} dx \Rightarrow du = x dx, v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\begin{aligned} \int x^3 \sqrt{1+x^2} dx &= \frac{1}{2}x^2 \left[\frac{2}{3}(1+x^2)^{3/2} \right] - \frac{2}{3} \int x(1+x^2)^{3/2} dx \\ &= \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{1}{2}(1+x^2)^{5/2} + C \\ &= \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2} + C \end{aligned}$$

Another method: Use substitution with $u = 1+x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.



42. First let $u = x^2$, $dv = \sin 2x dx \Rightarrow du = 2x dx$, $v = -\frac{1}{2} \cos 2x$.

Then $I = \int x^2 \sin 2x dx = -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x dx$.

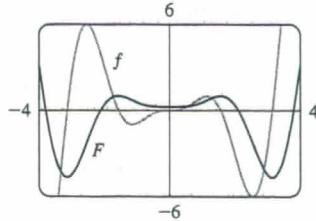
Next let $U = x$, $dV = \cos 2x dx \Rightarrow dU = dx$, $V = \frac{1}{2} \sin 2x$, so

$$\int x \cos 2x dx = \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x dx = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C.$$

$$\text{Thus, } I = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C.$$

We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative.

Note also that f is an odd function and F is an even function.



43. (a) Take $n = 2$ in Example 6 to get $\int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

$$(b) \int \sin^4 x dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{8} x - \frac{3}{16} \sin 2x + C.$$

44. (a) Let $u = \cos^{n-1} x$, $dv = \cos x dx \Rightarrow du = -(n-1) \cos^{n-2} x \sin x dx$, $v = \sin x$ in (2):

$$\begin{aligned} \int \cos^n x dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx \end{aligned}$$

Rearranging terms gives $n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$ or

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$(b) \text{Take } n = 2 \text{ in part (a) to get } \int \cos^2 x dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1 dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

$$(c) \int \cos^4 x dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x + C$$

45. (a) From Example 6, $\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$. Using (6),

$$\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \left[-\frac{\cos x \sin^{n-1} x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx \\ &= (0-0) + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx \end{aligned}$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x dx = \frac{2}{3} \int_0^{\pi/2} \sin x dx = \left[-\frac{2}{3} \cos x \right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n+1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$$\int_0^{\pi/2} \sin^{2k+1} x dx = \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{3 \cdot 5 \cdot 7 \cdots (2k+1)}. \text{ By Example 6,}$$

$$\begin{aligned} \int_0^{\pi/2} \sin^{2k+3} x dx &= \frac{2k+2}{2k+3} \int_0^{\pi/2} \sin^{2k+1} x dx = \frac{2k+2}{2k+3} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{3 \cdot 5 \cdot 7 \cdots (2k+1)} \\ &= \frac{2 \cdot 4 \cdot 6 \cdots (2k)[2(k+1)]}{3 \cdot 5 \cdot 7 \cdots (2k+1)[2(k+1)+1]}, \end{aligned}$$

so the formula holds for $n = k+1$. By induction, the formula holds for all $n \geq 1$.

46. Using Exercise 45(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} 1 \, dx = \frac{1}{2} [x]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{\pi}{2}$. By Exercise 45(a),

$$\begin{aligned}\int_0^{\pi/2} \sin^{2(k+1)} x \, dx &= \frac{2k+1}{2k+2} \int_0^{\pi/2} \sin^{2k} x \, dx = \frac{2k+1}{2k+2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \pi}{2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+2)} \cdot \frac{\pi}{2},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

47. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx.$$

48. Let $u = x^n$, $dv = e^x \, dx \Rightarrow du = nx^{n-1} \, dx$, $v = e^x$. By Equation 2, $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$.

49. $\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$
 $= I - \int \tan^{n-2} x \, dx$.

Let $u = \tan^{n-2} x$, $dv = \sec^2 x \, dx \Rightarrow du = (n-2) \tan^{n-3} x \sec^2 x \, dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned}I &= \tan^{n-1} x - (n-2) \int \tan^{n-2} x \sec^2 x \, dx \\ 1I &= \tan^{n-1} x - (n-2)I\end{aligned}$$

$$(n-1)I = \tan^{n-1} x$$

$$I = \frac{\tan^{n-1} x}{n-1}$$

$$\text{Returning to the original integral, } \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

50. Let $u = \sec^{n-2} x$, $dv = \sec^2 x \, dx \Rightarrow du = (n-2) \sec^{n-3} x \sec x \tan x \, dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned}\int \sec^n x \, dx &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \\ &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \tan x \sec^{n-2} x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx\end{aligned}$$

so $(n-1) \int \sec^n x \, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x \, dx$. If $n-1 \neq 0$, then

$$\int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

51. By repeated applications of the reduction formula in Exercise 47,

$$\begin{aligned}\int (\ln x)^3 \, dx &= x(\ln x)^3 - 3 \int (\ln x)^2 \, dx = x(\ln x)^3 - 3[x(\ln x)^2 - 2 \int (\ln x)^1 \, dx] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6[x(\ln x)^1 - 1 \int (\ln x)^0 \, dx] \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6 \int 1 \, dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C\end{aligned}$$

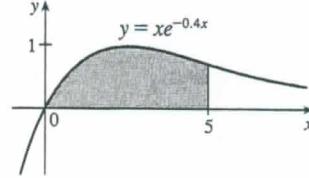
52. By repeated applications of the reduction formula in Exercise 48,

$$\begin{aligned}\int x^4 e^x dx &= x^4 e^x - 4 \int x^3 e^x dx = x^4 e^x - 4(x^3 e^x - 3 \int x^2 e^x dx) \\ &= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2 \int x^1 e^x dx) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(x^1 e^x - \int x^0 e^x dx) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C \quad [\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C]\end{aligned}$$

53. Area = $\int_0^5 xe^{-0.4x} dx$. Let $u = x$, $dv = e^{-0.4x} dx \Rightarrow$

$du = dx$, $v = -2.5e^{-0.4x}$. Then

$$\begin{aligned}\text{area} &= [-2.5xe^{-0.4x}]_0^5 + 2.5 \int_0^5 e^{-0.4x} dx \\ &= -12.5e^{-2} + 0 + 2.5[-2.5e^{-0.4x}]_0^5 \\ &= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \quad \text{or } \frac{25}{4} - \frac{75}{4}e^{-2}\end{aligned}$$



54. The curves $y = x \ln x$ and $y = 5 \ln x$ intersect when $x \ln x = 5 \ln x \Leftrightarrow x \ln x - 5 \ln x = 0 \Leftrightarrow (x-5) \ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5 \ln x > x \ln x$ since $\ln x > 0$. Thus,

$$\text{area} = \int_1^5 (5 \ln x - x \ln x) dx = \int_1^5 [(5-x) \ln x] dx. \text{ Let } u = \ln x, dv = (5-x) dx \Rightarrow du = dx/x, v = 5x - \frac{1}{2}x^2.$$

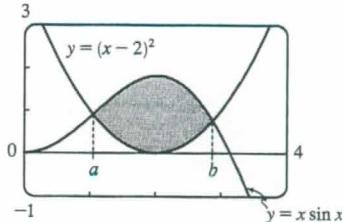
Then

$$\begin{aligned}\text{area} &= [(\ln x)(5x - \frac{1}{2}x^2)]_1^5 - \int_1^5 [(5x - \frac{1}{2}x^2)\frac{1}{x}] dx = (\ln 5)(\frac{25}{2}) - 0 - \int_1^5 (5 - \frac{1}{2}x) dx \\ &= \frac{25}{2} \ln 5 - [5x - \frac{1}{4}x^2]_1^5 = \frac{25}{2} \ln 5 - [(25 - \frac{25}{4}) - (5 - \frac{1}{4})] = \frac{25}{2} \ln 5 - 14\end{aligned}$$

55. The curves $y = x \sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and

$b \approx 2.87307$, so

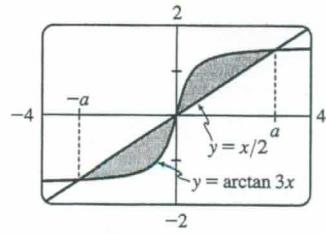
$$\begin{aligned}\text{area} &= \int_a^b [x \sin x - (x-2)^2] dx \\ &= [-x \cos x + \sin x - \frac{1}{3}(x-2)^3]_a^b \quad [\text{by Example 1}] \\ &\approx 2.81358 - 0.63075 = 2.18283\end{aligned}$$



56. The curves $y = \arctan 3x$ and $y = \frac{1}{2}x$ intersect at $x = \pm a \approx \pm 2.91379$,

so

$$\begin{aligned}\text{area} &= \int_{-a}^a |\arctan 3x - \frac{1}{2}x| dx = 2 \int_0^a (\arctan 3x - \frac{1}{2}x) dx \\ &= 2[x \arctan 3x - \frac{1}{6} \ln(1+9x^2) - \frac{1}{4}x^2]_0^a \quad [\text{see Example 5}] \\ &\approx 2(1.39768) = 2.79536\end{aligned}$$



57. $V = \int_0^1 2\pi x \cos(\pi x/2) dx$. Let $u = x$, $dv = \cos(\pi x/2) dx \Rightarrow du = dx$, $v = \frac{2}{\pi} \sin(\pi x/2)$.

$$V = 2\pi \left[\frac{2}{\pi} x \sin\left(\frac{\pi x}{2}\right) \right]_0^1 - 2\pi \cdot \frac{2}{\pi} \int_0^1 \sin\left(\frac{\pi x}{2}\right) dx = 2\pi \left(\frac{2}{\pi} - 0 \right) - 4 \left[-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_0^1 = 4 + \frac{8}{\pi}(0-1) = 4 - \frac{8}{\pi}.$$

58. Volume = $\int_0^1 2\pi x(e^x - e^{-x}) dx = 2\pi \int_0^1 (xe^x - xe^{-x}) dx = 2\pi \left[\int_0^1 xe^x dx - \int_0^1 xe^{-x} dx \right]$ [both integrals by parts]

$$= 2\pi \left[(xe^x - e^x) - (-xe^{-x} - e^{-x}) \right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$$

59. Volume = $\int_{-1}^0 2\pi(1-x)e^{-x} dx$. Let $u = 1-x$, $dv = e^{-x} dx \Rightarrow du = -dx$, $v = -e^{-x}$.

$$V = 2\pi[(1-x)(-e^{-x})]_{-1}^0 - 2\pi \int_{-1}^0 e^{-x} dx = 2\pi[(x-1)(e^{-x}) + e^{-x}]_{-1}^0 = 2\pi[xe^{-x}]_{-1}^0 = 2\pi(0+e) = 2\pi e$$

60. Volume = $\int_1^\pi 2\pi y \cdot \ln y dy = 2\pi[\frac{1}{2}y^2 \ln y - \frac{1}{4}y^2]_1^\pi$ [by parts with $u = \ln y$ and $dv = y dy$]

$$= 2\pi[\frac{1}{4}y^2(2\ln y - 1)]_1^\pi = 2\pi\left[\frac{\pi^2(2\ln \pi - 1)}{4} - \frac{(0-1)}{4}\right] = \pi^3 \ln \pi - \frac{\pi^3}{2} + \frac{\pi}{2}$$

61. The average value of $f(x) = x^2 \ln x$ on the interval $[1, 3]$ is $f_{ave} = \frac{1}{3-1} \int_1^3 x^2 \ln x dx = \frac{1}{2}I$.

Let $u = \ln x$, $dv = x^2 dx \Rightarrow du = (1/x) dx$, $v = \frac{1}{3}x^3$.

$$\text{So } I = [\frac{1}{3}x^3 \ln x]_1^3 - \int_1^3 \frac{1}{3}x^2 dx = (9 \ln 3 - 0) - [\frac{1}{9}x^3]_1^3 = 9 \ln 3 - (3 - \frac{1}{9}) = 9 \ln 3 - \frac{26}{9}.$$

$$\text{Thus, } f_{ave} = \frac{1}{2}I = \frac{1}{2}(9 \ln 3 - \frac{26}{9}) = \frac{9}{2} \ln 3 - \frac{13}{9}.$$

62. The rocket will have height $H = \int_0^{60} v(t) dt$ after 60 seconds.

$$\begin{aligned} H &= \int_0^{60} \left[-gt - v_e \ln\left(\frac{m-rt}{m}\right) \right] dt = -g[\frac{1}{2}t^2]_0^{60} - v_e \left[\int_0^{60} \ln(m-rt) dt - \int_0^{60} \ln m dt \right] \\ &= -g(1800) + v_e(\ln m)(60) - v_e \int_0^{60} \ln(m-rt) dt \end{aligned}$$

Let $u = \ln(m-rt)$, $dv = dt \Rightarrow du = \frac{1}{m-rt}(-r) dt$, $v = t$. Then

$$\begin{aligned} \int_0^{60} \ln(m-rt) dt &= [t \ln(m-rt)]_0^{60} + \int_0^{60} \frac{rt}{m-rt} dt = 60 \ln(m-60r) + \int_0^{60} \left(-1 + \frac{m}{m-rt} \right) dt \\ &= 60 \ln(m-60r) + \left[-t - \frac{m}{r} \ln(m-rt) \right]_0^{60} = 60 \ln(m-60r) - 60 - \frac{m}{r} \ln(m-60r) + \frac{m}{r} \ln m \end{aligned}$$

So $H = -1800g + 60v_e \ln m - 60v_e \ln(m-60r) + 60v_e + \frac{m}{r}v_e \ln(m-60r) - \frac{m}{r}v_e \ln m$. Substituting $g = 9.8$, $m = 30,000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14,844$ m.

63. Since $v(t) > 0$ for all t , the desired distance is $s(t) = \int_0^t v(w) dw = \int_0^t w^2 e^{-w} dw$.

First let $u = w^2$, $dv = e^{-w} dw \Rightarrow du = 2w dw$, $v = -e^{-w}$. Then $s(t) = [-w^2 e^{-w}]_0^t + 2 \int_0^t we^{-w} dw$.

Next let $U = w$, $dV = e^{-w} dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$\begin{aligned} s(t) &= -t^2 e^{-t} + 2 \left([-we^{-w}]_0^t + \int_0^t e^{-w} dw \right) = -t^2 e^{-t} + 2 \left(-te^{-t} + 0 + [-e^{-w}]_0^t \right) \\ &= -t^2 e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters} \end{aligned}$$

64. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x) dx \Rightarrow du = f'(x) dx$, $v = g'(x)$.

$$\text{Then } \int_0^a f(x) g''(x) dx = \left[f(x) g'(x) \right]_0^a - \int_0^a f'(x) g'(x) dx = f(a) g'(a) - \int_0^a f'(x) g'(x) dx.$$

Now let $U = f'(x)$, $dV = g'(x) dx \Rightarrow dU = f''(x) dx$ and $V = g(x)$, so

$$\int_0^a f'(x) g'(x) dx = \left[f'(x) g(x) \right]_0^a - \int_0^a f''(x) g(x) dx = f'(a) g(a) - \int_0^a f''(x) g(x) dx.$$

Combining the two results, we get $\int_0^a f(x) g''(x) dx = f(a) g'(a) - f'(a) g(a) + \int_0^a f''(x) g(x) dx$.

65. For $I = \int_1^4 xf''(x) dx$, let $u = x$, $dv = f''(x) dx \Rightarrow du = dx$, $v = f'(x)$. Then

$$I = [xf'(x)]_1^4 - \int_1^4 f'(x) dx = 4f'(4) - 1 \cdot f'(1) - [f(4) - f(1)] = 4 \cdot 3 - 1 \cdot 5 - (7 - 2) = 12 - 5 - 5 = 2.$$

We used the fact that f'' is continuous to guarantee that I exists.

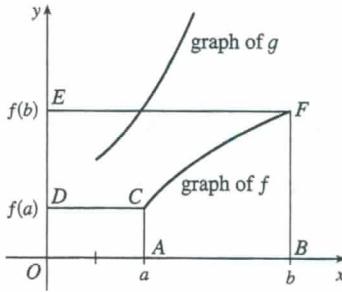
66. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x) dx = bf(b) - af(a) - \int_a^b x f'(x) dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x) dx$.

Then $\int_a^b x f'(x) dx = \int_{f(a)}^{f(b)} g(y) dy$. The result follows.

- (c) Part (b) says that the area of region $ABFC$ is

$$\begin{aligned} &= bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) dy \\ &= (\text{area of rectangle } OBFE) - (\text{area of rectangle } OACD) - (\text{area of region } DCFE) \end{aligned}$$



- (d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x dx = e \ln e - 1 \ln 1 - \int_{\ln 1}^{\ln e} e^y dy = e - \int_0^1 e^y dy = e - [e^y]_0^1 = e - (e - 1) = 1.$$

67. Using the formula for volumes of rotation and the figure, we see that

$$\text{Volume} = \int_0^d \pi b^2 dy - \int_0^c \pi a^2 dy - \int_c^d \pi [g(y)]^2 dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2 dy. \text{ Let } y = f(x),$$

which gives $dy = f'(x) dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x) dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x) dx \Rightarrow du = 2x dx$, $v = f(x)$, and

$$\int_a^b x^2 f'(x) dx = [x^2 f(x)]_a^b - \int_a^b 2x f(x) dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x f(x) dx, \text{ but } f(a) = c \text{ and } f(b) = d \Rightarrow$$

$$V = \pi b^2 d - \pi a^2 c - \pi \left[b^2 d - a^2 c - \int_a^b 2x f(x) dx \right] = \int_a^b 2\pi x f(x) dx.$$

68. (a) We note that for $0 \leq x \leq \frac{\pi}{2}$, $0 \leq \sin x \leq 1$, so $\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$.

- (b) Substituting directly into the result from Exercise 46, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\frac{1 \cdot 3 \cdot 5 \cdots [2(n+1)-1] \pi}{2 \cdot 4 \cdot 6 \cdots [2(n+1)]} \frac{1}{2}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{1}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n} . The inequalities are preserved since I_{2n} is positive: $\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n}}{I_{2n}}$.

Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes $\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$. Now

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} = \lim_{n \rightarrow \infty} 1 = 1, \text{ so by the Squeeze Theorem, } \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

(d) We substitute the results from Exercises 45 and 46 into the result from part (c):

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n \rightarrow \infty} \frac{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)}} = \lim_{n \rightarrow \infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right] \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1) (\pi)} \left(\frac{2}{\pi} \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi} \quad [\text{rearrange terms}] \end{aligned}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the Wallis product:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

(e) The area of the k th rectangle is k . At the $2n$ th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by

$\frac{2n}{2n-1}$, and at the $(2n+1)$ th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\frac{2n+1}{2n}$. These

two steps multiply the ratio of width to height by $\frac{2n}{2n-1}$ and $\frac{1}{(2n+1)/(2n)} = \frac{2n}{2n+1}$ respectively. So, by part (d), the

limiting ratio is $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{\pi}{2}$.

7.2 Trigonometric Integrals

The symbols $\stackrel{s}{=}$ and $\stackrel{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x dx\}$ and $\{u = \cos x, du = -\sin x dx\}$, respectively.

$$\begin{aligned} 1. \int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx \stackrel{c}{=} \int (1 - u^2) u^2 (-du) \\ &= \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \end{aligned}$$

$$\begin{aligned} 2. \int \sin^6 x \cos^3 x dx &= \int \sin^6 x \cos^2 x \cos x dx = \int \sin^6 x (1 - \sin^2 x) \cos x dx \stackrel{s}{=} \int u^6 (1 - u^2) du \\ &= \int (u^6 - u^8) du = \frac{1}{7} u^7 - \frac{1}{9} u^9 + C = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C \end{aligned}$$

$$\begin{aligned} 3. \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x dx &= \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x dx = \int_{\pi/2}^{3\pi/4} \sin^5 x (1 - \sin^2 x) \cos x dx \stackrel{s}{=} \int_1^{\sqrt{2}/2} u^5 (1 - u^2) du \\ &= \int_1^{\sqrt{2}/2} (u^5 - u^7) du = \left[\frac{1}{6} u^6 - \frac{1}{8} u^8 \right]_1^{\sqrt{2}/2} = \left(\frac{1/8}{6} - \frac{1/16}{8} \right) - \left(\frac{1}{6} - \frac{1}{8} \right) = -\frac{11}{384} \end{aligned}$$

$$\begin{aligned} 4. \int_0^{\pi/2} \cos^5 x dx &= \int_0^{\pi/2} (\cos^2 x)^2 \cos x dx = \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x dx \stackrel{s}{=} \int_0^1 (1 - u^2)^2 du \\ &= \int_0^1 (1 - 2u^2 + u^4) du = \left[u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{8}{15} \end{aligned}$$