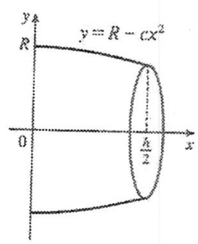
71. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c\left(\frac{1}{2}h\right)^2 = R - d = r$.



(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi y^2\,dx = 2\pi\int_0^{h/2}\left(R - cx^2\right)^2 dx = 2\pi\left[R^2x - \tfrac{2}{3}Rcx^3 + \tfrac{1}{5}c^2x^5\right]_0^{h/2}$$
$$= 2\pi\left(\tfrac{1}{2}R^2h - \tfrac{1}{12}Rch^3 + \tfrac{1}{160}c^2h^5\right)$$

Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h\left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4\right)\right]$.

But $R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = \left(R - \frac{1}{4}ch^2\right)^2 - \frac{1}{40}c^2h^4 = (R-d)^2 - \frac{2}{5}\left(\frac{1}{4}ch^2\right)^2 = r^2 - \frac{2}{5}d^2$.

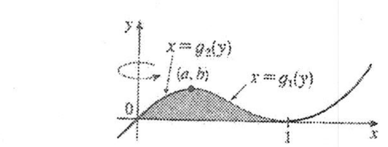
Substituting this back into V, we see that $V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$, as required.

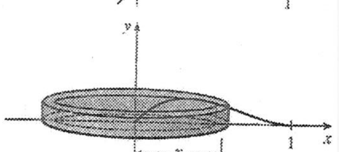
72. It suffices to consider the case where $\mathcal{R}$ is bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$, where $g(x) \le f(x)$ for all x in $[a, b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when $\mathcal{R}$ is rotated about the line $y = -k$, which is equal to

$$V_2 = \pi\int_a^b\left([f(x)+k]^2 - [g(x)+k]^2\right)dx = \pi\int_a^b\left([f(x)]^2 - [g(x)]^2\right)dx + 2\pi k\int_a^b[f(x) - g(x)]\,dx = V_1 + 2\pi kA$$

6.3 Volumes by Cylindrical Shells

1.



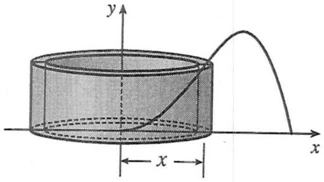


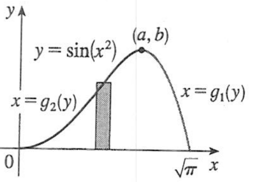
If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi\int_0^b\left\{[g_1(y)]^2 - [g_2(y)]^2\right\}dy$.

Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x\left[x(x-1)^2\right]dx = 2\pi\int_0^1\left(x^4 - 2x^3 + x^2\right)dx = 2\pi\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

2.

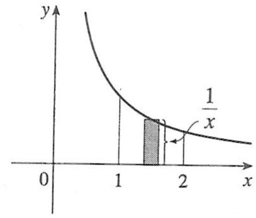


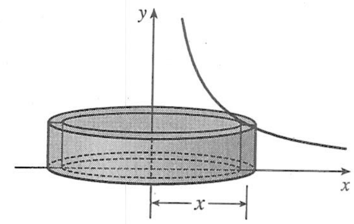


A typical cylindrical shell has circumference $2\pi x$ and height $\sin(x^2)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx$. Let $u = x^2$. Then $du = 2x\,dx$, so $V = \pi \int_0^{\pi} \sin u\,du = \pi[-\cos u]_0^{\pi} = \pi[1 - (-1)] = 2\pi$. For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using $V = \pi \int_0^b \{[g_1(y)]^2 - [g_2(y)]^2\}\,dy$. Using shells is definitely preferable to slicing.

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\,dx = 2\pi \int_1^2 1\,dx$

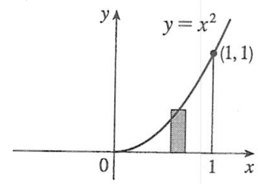
$= 2\pi\left[x\right]_1^2 = 2\pi(2-1) = 2\pi$

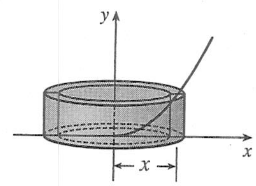




4. $V = \int_0^1 2\pi x \cdot x^2\,dx = 2\pi \int_0^1 x^3\,dx$

$= 2\pi\left[\frac{1}{4}x^4\right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$

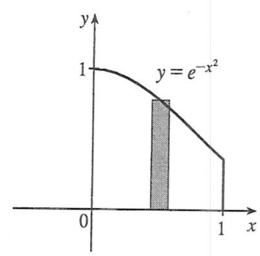


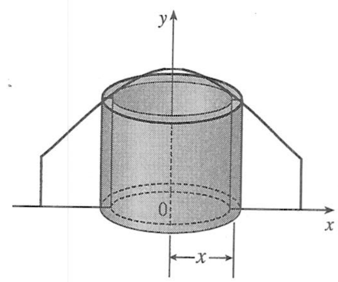


5. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$.

Thus, $du = 2x\,dx$, so

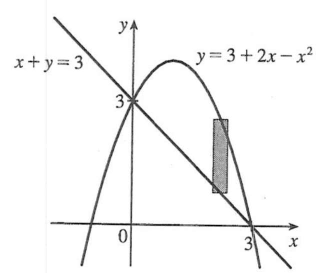
$V = \pi \int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

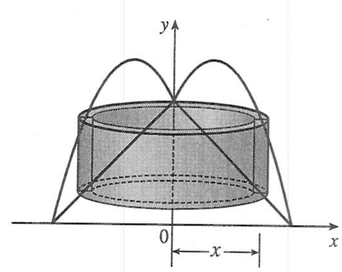




6. $V = 2\pi \int_0^3 \{x[(3 + 2x - x^2) - (3 - x)]\}\,dx = 2\pi \int_0^3 [x(3x - x^2)]\,dx$

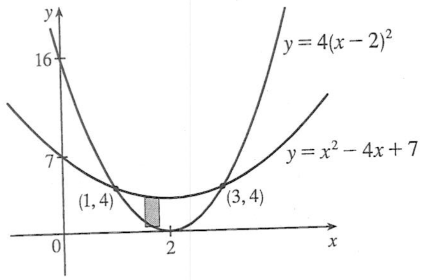
$= 2\pi \int_0^3 (3x^2 - x^3)\,dx = 2\pi\left[x^3 - \frac{1}{4}x^4\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27}{2}\pi$

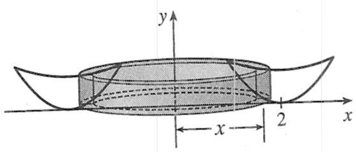




7. The curves intersect when $4(x-2)^2 = x^2 - 4x + 7 \;\Leftrightarrow\; 4x^2 - 16x + 16 = x^2 - 4x + 7 \;\Leftrightarrow$
$3x^2 - 12x + 9 = 0 \;\Leftrightarrow\; 3(x^2 - 4x + 3) = 0 \;\Leftrightarrow\; 3(x-1)(x-3) = 0$, so $x = 1$ or 3.

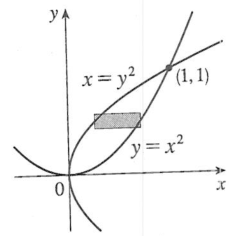
$$V = 2\pi \int_1^3 \left\{x\left[(x^2 - 4x + 7) - 4(x-2)^2\right]\right\} dx = 2\pi \int_1^3 \left[x(x^2 - 4x + 7 - 4x^2 + 16x - 16)\right] dx$$
$$= 2\pi \int_1^3 \left[x(-3x^2 + 12x - 9)\right] dx = 2\pi(-3)\int_1^3 (x^3 - 4x^2 + 3x)\, dx = -6\pi\left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_1^3$$
$$= -6\pi\left[\left(\tfrac{81}{4} - 36 + \tfrac{27}{2}\right) - \left(\tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \tfrac{4}{3}\right) = -6\pi\left(-\tfrac{8}{3}\right) = 16\pi$$

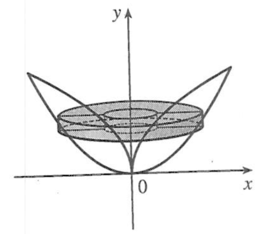




8. By slicing:

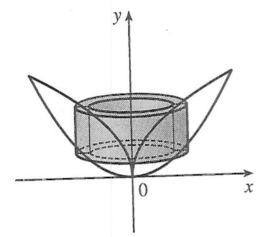
$$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - (y^2)^2\right] dy = \pi\int_0^1 (y - y^4)\, dy$$
$$= \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \tfrac{3}{10}\pi$$



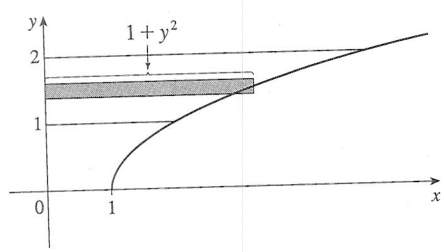


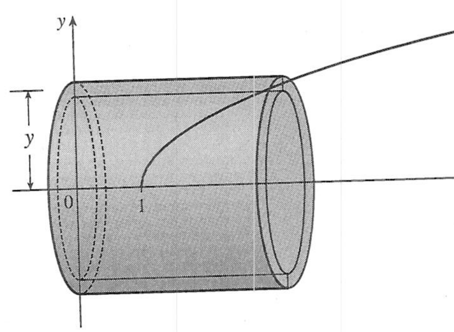
By cylindrical shells:

$$V = \int_0^1 2\pi x\left(\sqrt{x} - x^2\right) dx = 2\pi\int_0^1 (x^{3/2} - x^3)\, dx = 2\pi\left[\tfrac{2}{5}x^{5/2} - \tfrac{1}{4}x^4\right]_0^1$$
$$= 2\pi\left(\tfrac{2}{5} - \tfrac{1}{4}\right) = 2\pi\left(\tfrac{3}{20}\right) = \tfrac{3}{10}\pi$$



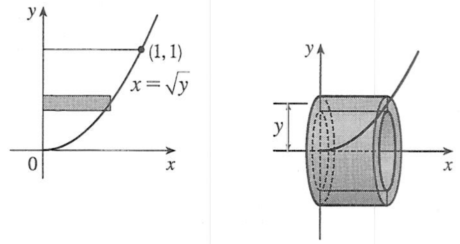
9. $V = \int_1^2 2\pi y(1 + y^2)\, dy = 2\pi\int_1^2 (y + y^3)\, dy = 2\pi\left[\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4\right]_1^2$
$= 2\pi\left[(2 + 4) - \left(\tfrac{1}{2} + \tfrac{1}{4}\right)\right] = 2\pi\left(\tfrac{21}{4}\right) = \tfrac{21}{2}\pi$





10. $V = \int_0^1 2\pi y \sqrt{y}\, dy = 2\pi \int_0^1 y^{3/2}\, dy$

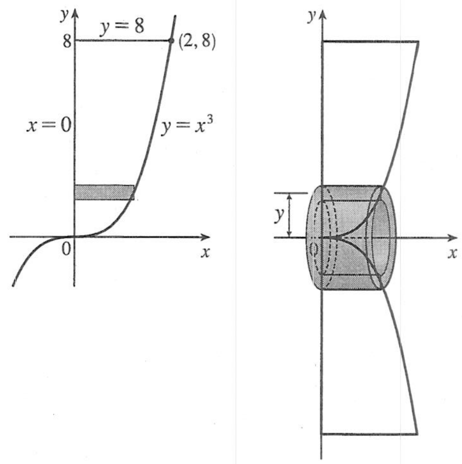
$= 2\pi \left[\frac{2}{5} y^{5/2}\right]_0^1 = \frac{4}{5}\pi$



11. $V = 2\pi \int_0^8 \left[y(\sqrt[3]{y} - 0)\right] dy$

$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \left[\frac{3}{7} y^{7/3}\right]_0^8$

$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768}{7}\pi$

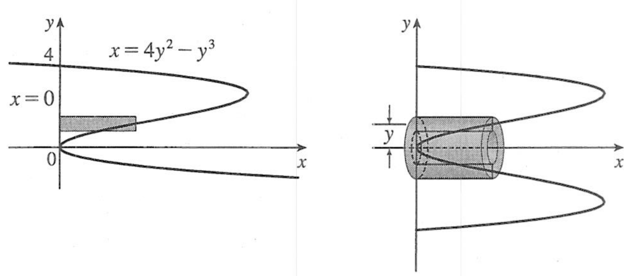


12. $V = 2\pi \int_0^4 \left[y(4y^2 - y^3)\right] dy$

$= 2\pi \int_0^4 (4y^3 - y^4)\, dy$

$= 2\pi \left[y^4 - \frac{1}{5} y^5\right]_0^4 = 2\pi \left(256 - \frac{1024}{5}\right)$

$= 2\pi \left(\frac{256}{5}\right) = \frac{512}{5}\pi$



13. The height of the shell is $2 - \left[1 + (y - 2)^2\right] = 1 - (y - 2)^2 = 1 - \left(y^2 - 4y + 4\right) = -y^2 + 4y - 3$.

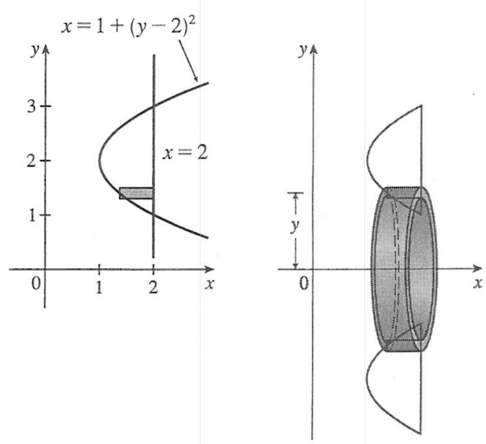
$V = 2\pi \int_1^3 y(-y^2 + 4y - 3)\, dy$

$= 2\pi \int_1^3 (-y^3 + 4y^2 - 3y)\, dy$

$= 2\pi \left[-\frac{1}{4} y^4 + \frac{4}{3} y^3 - \frac{3}{2} y^2\right]_1^3$

$= 2\pi \left[\left(-\frac{81}{4} + 36 - \frac{27}{2}\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right]$

$= 2\pi \left(\frac{8}{3}\right) = \frac{16}{3}\pi$

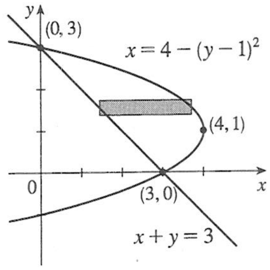


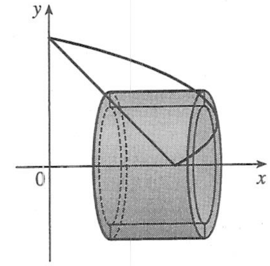
14. $V = \int_0^3 2\pi y\left[4 - (y-1)^2 - (3-y)\right] dy$

$= 2\pi \int_0^3 y(-y^2 + 3y)\, dy$

$= 2\pi \int_0^3 (-y^3 + 3y^2)\, dy = 2\pi\left[-\frac{1}{4}y^4 + y^3\right]_0^3$

$= 2\pi\left(-\frac{81}{4} + 27\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27}{2}\pi$





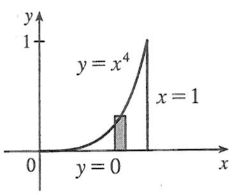
15. The shell has radius $2 - x$, circumference $2\pi(2-x)$, and height x^4.

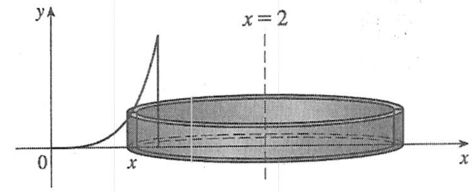
$V = \int_0^1 2\pi(2-x)x^4\, dx$

$= 2\pi \int_0^1 (2x^4 - x^5)\, dx$

$= 2\pi\left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right]_0^1$

$= 2\pi\left[\left(\frac{2}{5} - \frac{1}{6}\right) - 0\right] = 2\pi\left(\frac{7}{30}\right) = \frac{7}{15}\pi$





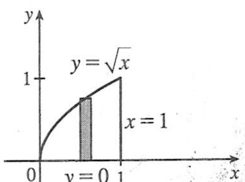
16. The shell has radius $x - (-1) = x + 1$, circumference $2\pi(x+1)$, and height $\sqrt{x}$.

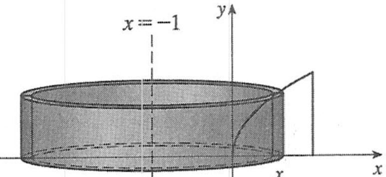
$V = \int_0^1 2\pi(x+1)\sqrt{x}\, dx$

$= 2\pi \int_0^1 (x^{3/2} + x^{1/2})\, dx$

$= 2\pi\left[\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right]_0^1$

$= 2\pi\left[\left(\frac{2}{5} + \frac{2}{3}\right) - 0\right] = 2\pi\left(\frac{16}{15}\right) = \frac{32}{15}\pi$





17. The shell has radius $x - 1$, circumference $2\pi(x-1)$, and height $(4x - x^2) - 3 = -x^2 + 4x - 3$.

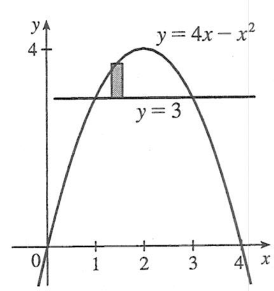
$V = \int_1^3 2\pi(x-1)(-x^2 + 4x - 3)\, dx$

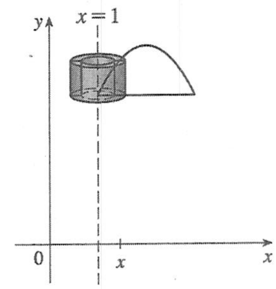
$= 2\pi \int_1^3 (-x^3 + 5x^2 - 7x + 3)\, dx$

$= 2\pi\left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - \frac{7}{2}x^2 + 3x\right]_1^3$

$= 2\pi\left[\left(-\frac{81}{4} + 45 - \frac{63}{2} + 9\right) - \left(-\frac{1}{4} + \frac{5}{3} - \frac{7}{2} + 3\right)\right]$

$= 2\pi\left(\frac{4}{3}\right) = \frac{8}{3}\pi$





18. The shell has radius $1 - x$, circumference $2\pi(1-x)$, and height $(2 - x^2) - x^2 = 2 - 2x^2$.

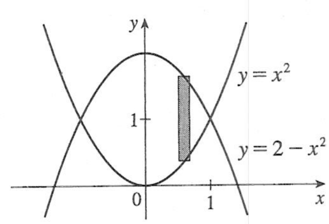
$V = \int_{-1}^1 2\pi(1-x)(2 - 2x^2)\, dx$

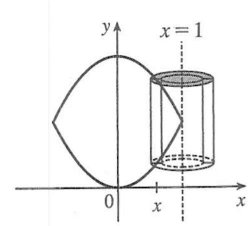
$= 2\pi(2) \int_{-1}^1 (1-x)(1-x^2)\, dx$

$= 4\pi \int_{-1}^1 (1 - x - x^2 + x^3)\, dx$

$= 4\pi(2) \int_0^1 (1 - x^2)\, dx$ [by Theorem 5.5.7]

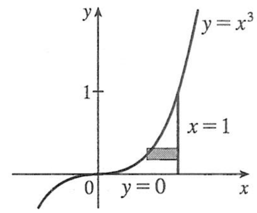
$= 8\pi\left[x - \frac{1}{3}x^3\right]_0^1 = 8\pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = 8\pi\left(\frac{2}{3}\right) = \frac{16}{3}\pi$

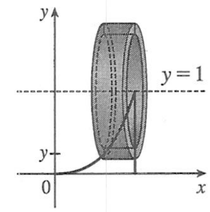




19. The shell has radius $1-y$, circumference $2\pi(1-y)$, and height $1-\sqrt[3]{y}$ $\left[y=x^3 \;\Leftrightarrow\; x=\sqrt[3]{y}\right]$.

$$\begin{aligned} V &= \int_0^1 2\pi(1-y)(1-y^{1/3})\,dy \\ &= 2\pi\int_0^1 (1-y-y^{1/3}+y^{4/3})\,dy \\ &= 2\pi\left[y-\tfrac{1}{2}y^2-\tfrac{3}{4}y^{4/3}+\tfrac{3}{7}y^{7/3}\right]_0^1 \\ &= 2\pi\left[\left(1-\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{3}{7}\right)-0\right] \\ &= 2\pi\left(\tfrac{5}{28}\right)=\tfrac{5}{14}\pi \end{aligned}$$

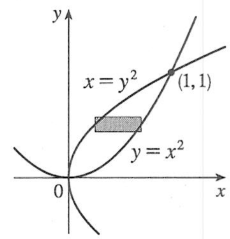


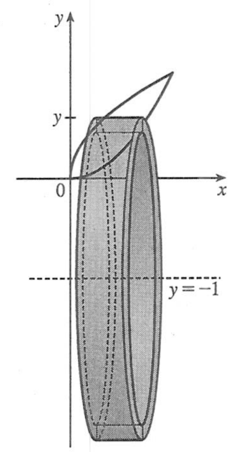


20. The shell has radius $y-(-1)=y+1$,

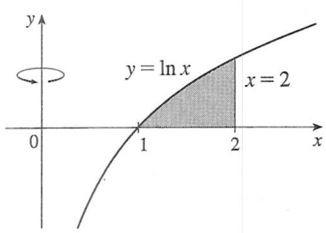
circumference $2\pi(y+1)$, and height $\sqrt{y}-y^2$.

$$\begin{aligned} V &= \int_0^1 2\pi(y+1)\left(\sqrt{y}-y^2\right)dy \\ &= 2\pi\int_0^1 (y^{3/2}+y^{1/2}-y^3-y^2)\,dy \\ &= 2\pi\left[\tfrac{2}{5}y^{5/2}+\tfrac{2}{3}y^{3/2}-\tfrac{1}{4}y^4-\tfrac{1}{3}y^3\right]_0^1 \\ &= 2\pi\left(\tfrac{2}{5}+\tfrac{2}{3}-\tfrac{1}{4}-\tfrac{1}{3}\right)=2\pi\left(\tfrac{29}{60}\right)=\tfrac{29}{30}\pi \end{aligned}$$

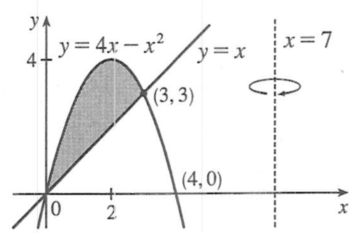




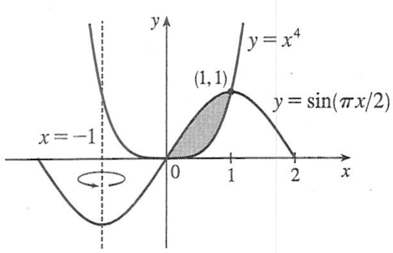
21. $V=\int_1^2 2\pi x\ln x\,dx$



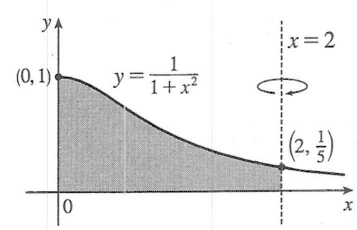
22. $V=\int_0^3 2\pi(7-x)[(4x-x^2)-x]\,dx$



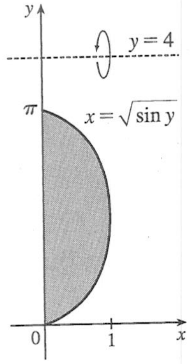
23. $V=\int_0^1 2\pi[x-(-1)]\left(\sin\frac{\pi}{2}x-x^4\right)dx$



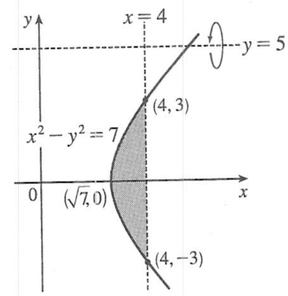
24. $V=\displaystyle\int_0^2 2\pi(2-x)\left(\frac{1}{1+x^2}\right)dx$



25. $V = \int_0^{\pi} 2\pi(4-y)\sqrt{\sin y}\,dy$



26. $V = \int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{y^2+7}\right)dy$

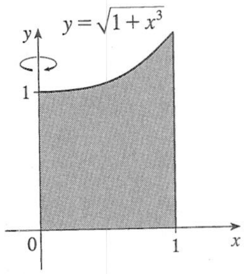


27. $V = \int_0^1 2\pi x\sqrt{1+x^3}\,dx$. Let $f(x) = x\sqrt{1+x^3}$.

Then the Midpoint Rule with $n = 5$ gives

$$\int_0^1 f(x)\,dx \approx \tfrac{1-0}{5}\left[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)\right]$$
$$\approx 0.2(2.9290)$$

Multiplying by 2π gives $V \approx 3.68$.



28. $\Delta x = \frac{12-2}{5} = 2$, $n = 5$ and $x_i^* = 2 + (2i+1)$, where $i = 0, 1, 2, 3, 4$. The values of $f(x)$ are taken directly from the diagram.

$$V = \int_2^{12} 2\pi x f(x)\,dx \approx 2\pi[3f(3)+5f(5)+7f(7)+9f(9)+11f(11)]\cdot 2$$
$$\approx 2\pi[3(2)+5(4)+7(4)+9(2)+11(1)]2 = 332\pi$$

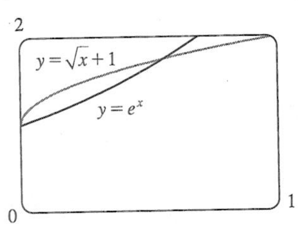
29. $\int_0^3 2\pi x^5\,dx = 2\pi\int_0^3 x(x^4)\,dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

30. $2\pi\displaystyle\int_0^2 \frac{y}{1+y^2}\,dy = 2\pi\int_0^2 y\left(\frac{1}{1+y^2}\right)dy$. The solid is obtained by rotating the region $0 \le x \le \dfrac{1}{1+y^2}$, $0 \le y \le 2$ about the x-axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2)\,dy$. The solid is obtained by rotating the region bounded by (i) $x = 1-y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\,dx$. The solid is obtained by rotating the region bounded by (i) $0 \le y \le \cos x - \sin x$, $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$, $0 \le x \le \frac{\pi}{4}$ about the line $x = \pi$ using cylindrical shells.

33.



From the graph, the curves intersect at $x = 0$ and $x = a \approx 0.56$, with $\sqrt{x}+1 > e^x$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi\int_0^a x\left[\left(\sqrt{x}+1\right) - e^x\right]dx \approx 0.13.$$