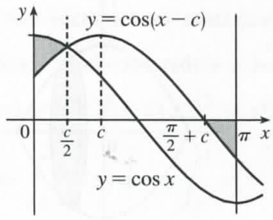
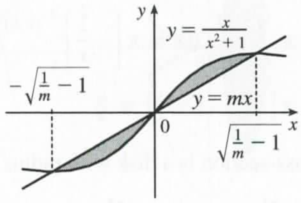
52. 

It appears from the diagram that the curves $y = \cos x$ and $y = \cos(x - c)$ intersect halfway between 0 and c, namely, when $x = c/2$. We can verify that this is indeed true by noting that $\cos(c/2 - c) = \cos(-c/2) = \cos(c/2)$. The point where $\cos(x - c)$ crosses the x-axis is $x = \frac{\pi}{2} + c$. So we require that $\int_0^{c/2} [\cos x - \cos(x - c)]\, dx = -\int_{\pi/2+c}^{\pi} \cos(x - c)\, dx$ [the negative sign on the RHS is needed since the second area is beneath the x-axis] $\Leftrightarrow$ $[\sin x - \sin(x - c)]_0^{c/2} = -[\sin(x - c)]_{\pi/2+c}^{\pi}$ $\Rightarrow$ $[\sin(c/2) - \sin(-c/2)] - [-\sin(-c)] = -\sin(\pi - c) + \sin\left[\left(\frac{\pi}{2} + c\right) - c\right]$ $\Leftrightarrow$ $2\sin(c/2) - \sin c = -\sin c + 1$. [Here we have used the oddness of the sine function, and the fact that $\sin(\pi - c) = \sin c$]. So $2\sin(c/2) = 1$ $\Leftrightarrow$ $\sin(c/2) = \frac{1}{2}$ $\Leftrightarrow$ $c/2 = \frac{\pi}{6}$ $\Leftrightarrow$ $c = \frac{\pi}{3}$.

53. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2 + 1) = mx$ $\Rightarrow$ $x = x(mx^2 + m)$ $\Rightarrow$ $x(mx^2 + m) - x = 0$ $\Rightarrow$ $x(mx^2 + m - 1) = 0$ $\Rightarrow$ $x = 0$ or $mx^2 + m - 1 = 0$ $\Rightarrow$



$x = 0$ or $x^2 = \dfrac{1 - m}{m}$ $\Rightarrow$ $x = 0$ or $x = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0$ $\Leftrightarrow$ $1/m > 1$ $\Leftrightarrow$ $0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2 + 1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

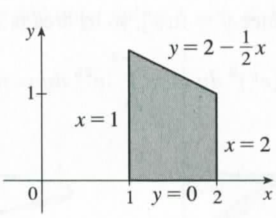
$$2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2 + 1} - mx\right] dx = 2\left[\tfrac{1}{2}\ln(x^2 + 1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} = [\ln(1/m - 1 + 1) - m(1/m - 1)] - (\ln 1 - 0)$$

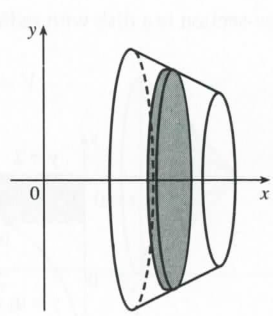
$$= \ln(1/m) - 1 + m = m - \ln m - 1$$

6.2 Volumes

1. A cross-section is a disk with radius $2 - \frac{1}{2}x$, so its area is $A(x) = \pi\left(2 - \frac{1}{2}x\right)^2$.

$$V = \int_1^2 A(x)\, dx = \int_1^2 \pi\left(2 - \tfrac{1}{2}x\right)^2 dx$$
$$= \pi \int_1^2 \left(4 - 2x + \tfrac{1}{4}x^2\right) dx$$
$$= \pi\left[4x - x^2 + \tfrac{1}{12}x^3\right]_1^2$$
$$= \pi\left[\left(8 - 4 + \tfrac{8}{12}\right) - \left(4 - 1 + \tfrac{1}{12}\right)\right]$$
$$= \pi\left(1 + \tfrac{7}{12}\right) = \tfrac{19}{12}\pi$$





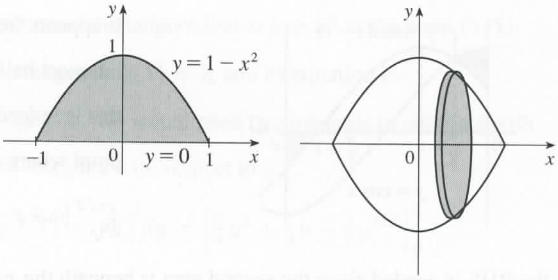
2. A cross-section is a disk with radius $1 - x^2$, so its area is

$A(x) = \pi(1 - x^2)^2$.

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(1 - x^2)^2\,dx$$

$$= 2\pi \int_0^1 (1 - 2x^2 + x^4)\,dx = 2\pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1$$

$$= 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 2\pi\left(\tfrac{8}{15}\right) = \tfrac{16}{15}\pi$$



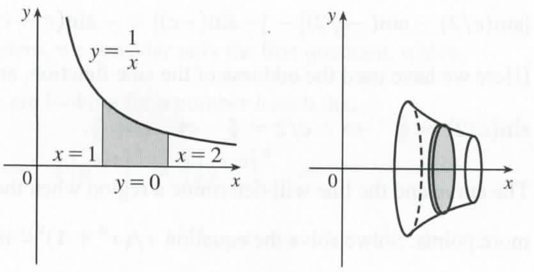
3. A cross-section is a disk with radius $1/x$, so its area is

$A(x) = \pi(1/x)^2$.

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2$$

$$= \pi\left[-\tfrac{1}{2} - (-1)\right] = \tfrac{\pi}{2}$$

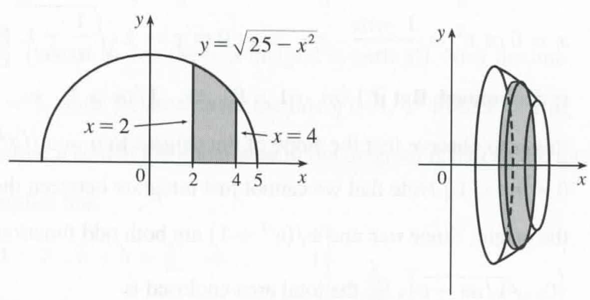


4. A cross-section is a disk with radius $\sqrt{25 - x^2}$, so its area is $A(x) = \pi\left(\sqrt{25 - x^2}\right)^2$.

$$V = \int_2^4 A(x)\,dx = \int_2^4 \pi\left(\sqrt{25 - x^2}\right)^2 dx$$

$$= \pi \int_2^4 (25 - x^2)\,dx = \pi\left[25x - \tfrac{1}{3}x^3\right]_2^4$$

$$= \pi\left[\left(100 - \tfrac{64}{3}\right) - \left(50 - \tfrac{8}{3}\right)\right] = \tfrac{94}{3}\pi$$

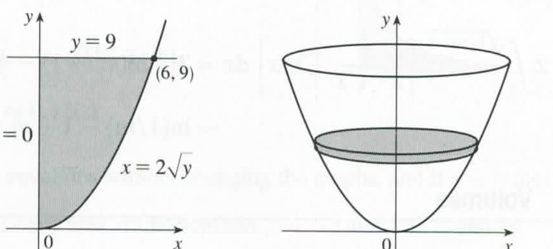


5. A cross-section is a disk with radius $2\sqrt{y}$, so its area is

$A(y) = \pi\left(2\sqrt{y}\right)^2$.

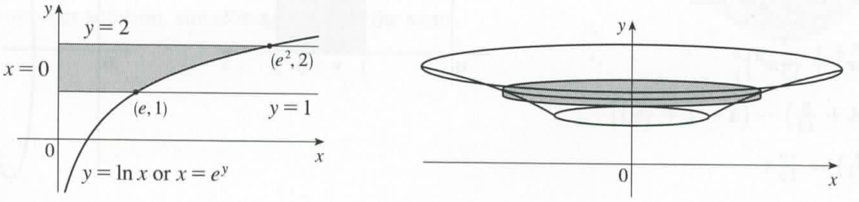
$$V = \int_0^9 A(y)\,dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy = 4\pi \int_0^9 y\,dy$$

$$= 4\pi\left[\tfrac{1}{2}y^2\right]_0^9 = 2\pi(81) = 162\pi$$



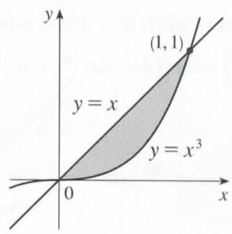
6. A cross-section is a disk with radius e^y [since $y = \ln x$], so its area is $A(y) = \pi(e^y)^2$.

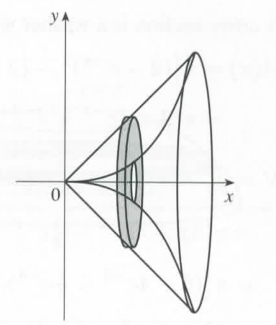
$$V = \int_1^2 \pi(e^y)^2\,dy = \pi \int_1^2 e^{2y}\,dy = \pi\left[\frac{1}{2}e^{2y}\right]_1^2 = \frac{\pi}{2}(e^4 - e^2)$$



7. A cross-section is a washer (annulus) with inner radius x^3 and outer radius x, so its area is $A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$.

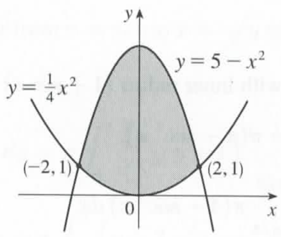
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx$$
$$= \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{7}\right) = \tfrac{4}{21}\pi$$

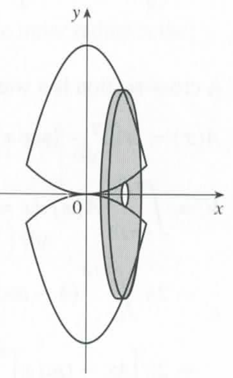




8. A cross-section is a washer with inner radius $\frac{1}{4}x^2$ and outer radius $5 - x^2$, so its area is

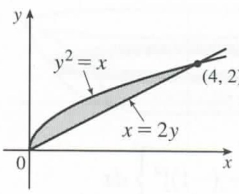
$$A(x) = \pi(5 - x^2)^2 - \pi\left(\tfrac{1}{4}x^2\right)^2$$
$$= \pi\left(25 - 10x^2 + x^4 - \tfrac{1}{16}x^4\right).$$
$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \pi\left(25 - 10x^2 + \tfrac{15}{16}x^4\right) dx$$
$$= 2\pi \int_0^2 \left(25 - 10x^2 + \tfrac{15}{16}x^4\right) dx$$
$$= 2\pi\left[25x - \tfrac{10}{3}x^3 + \tfrac{3}{16}x^5\right]_0^2 = 2\pi\left(50 - \tfrac{80}{3} + 6\right) = \tfrac{176}{3}\pi$$

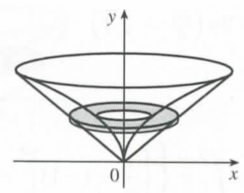




9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is $A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4)$.

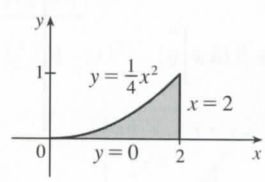
$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 (4y^2 - y^4)\,dy$$
$$= \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64}{15}\pi$$

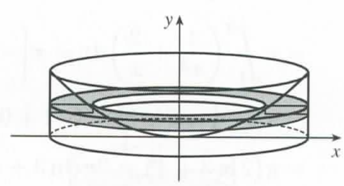




10. A cross-section is a washer with inner radius $x = 2\sqrt{y}$ and outer radius 2, so its area is

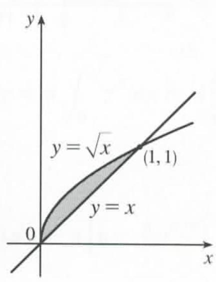
$$A(y) = \pi\left[(2)^2 - \left(2\sqrt{y}\right)^2\right]$$
$$= \pi(4 - 4y) = 4\pi(1 - y).$$
$$V = \int_0^1 A(y)\,dy = \int_0^1 4\pi(1 - y)\,dy$$
$$= 4\pi\left[y - \tfrac{1}{2}y^2\right]_0^1 = 4\pi\left[\left(1 - \tfrac{1}{2}\right) - 0\right] = 2\pi$$

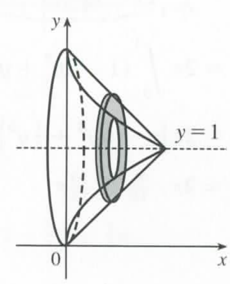




11. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

$$A(x) = \pi(1 - x)^2 - \pi\left(1 - \sqrt{x}\right)^2$$
$$= \pi\left[(1 - 2x + x^2) - \left(1 - 2\sqrt{x} + x\right)\right]$$
$$= \pi\left(-3x + x^2 + 2\sqrt{x}\right).$$
$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 \left(-3x + x^2 + 2\sqrt{x}\right) dx$$
$$= \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$

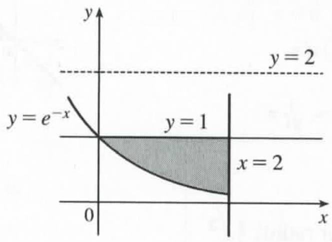


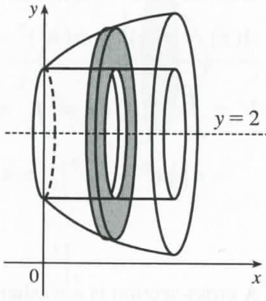


12. A cross-section is a washer with inner radius $2-1$ and outer radius $2-e^{-x}$, so its area is

$$A(x)=\pi\left[(2-e^{-x})^2-(2-1)^2\right]=\pi\left[(4-4e^{-x}+e^{-2x})-1\right]$$
$$=\pi(3-4e^{-x}+e^{-2x}).$$

$$V=\int_0^2 A(x)\,dx=\int_0^2 \pi(3-4e^{-x}+e^{-2x})\,dx$$
$$=\pi\left[3x+4e^{-x}-\tfrac{1}{2}e^{-2x}\right]_0^2$$
$$=\pi\left[(6+4e^{-2}-\tfrac{1}{2}e^{-4})-(0+4-\tfrac{1}{2})\right]$$
$$=\left(\tfrac{5}{2}+4e^{-2}-\tfrac{1}{2}e^{-4}\right)\pi$$

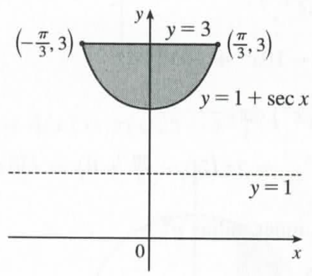


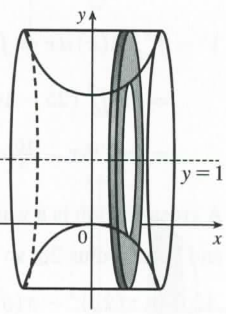


13. A cross-section is a washer with inner radius $(1+\sec x)-1=\sec x$ and outer radius $3-1=2$, so its area is

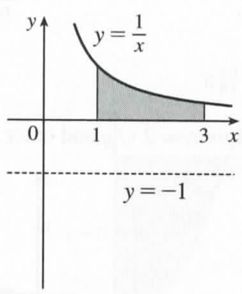
$$A(x)=\pi\left[2^2-(\sec x)^2\right]=\pi(4-\sec^2 x).$$

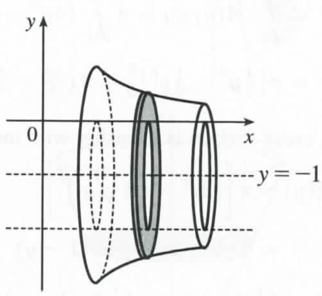
$$V=\int_{-\pi/3}^{\pi/3} A(x)\,dx=\int_{-\pi/3}^{\pi/3}\pi(4-\sec^2 x)\,dx$$
$$=2\pi\int_0^{\pi/3}(4-\sec^2 x)\,dx \qquad \text{[by symmetry]}$$
$$=2\pi\Big[4x-\tan x\Big]_0^{\pi/3}=2\pi\left[\left(\tfrac{4\pi}{3}-\sqrt{3}\right)-0\right]$$
$$=2\pi\left(\tfrac{4\pi}{3}-\sqrt{3}\right)$$



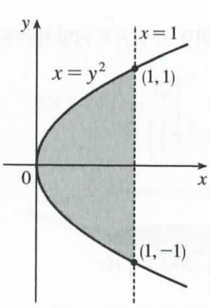


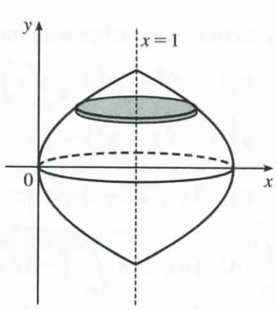
14. $$V=\int_1^3 \pi\left\{\left[\frac{1}{x}-(-1)\right]^2-[0-(-1)]^2\right\}dx$$
$$=\pi\int_1^3\left[\left(\frac{1}{x}+1\right)^2-1^2\right]dx$$
$$=\pi\int_1^3\left(\frac{1}{x^2}+\frac{2}{x}\right)dx=\pi\left[-\frac{1}{x}+2\ln x\right]_1^3$$
$$=\pi\left[\left(-\tfrac{1}{3}+2\ln 3\right)-(-1+0)\right]$$
$$=\pi\left(2\ln 3+\tfrac{2}{3}\right)=2\pi\left(\ln 3+\tfrac{1}{3}\right)$$





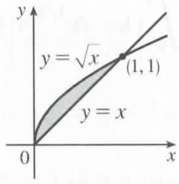
15. $$V=\int_{-1}^1 \pi(1-y^2)^2\,dy=2\int_0^1 \pi(1-y^2)^2\,dy$$
$$=2\pi\int_0^1(1-2y^2+y^4)\,dy$$
$$=2\pi\left[y-\tfrac{2}{3}y^3+\tfrac{1}{5}y^5\right]_0^1$$
$$=2\pi\cdot\tfrac{8}{15}=\tfrac{16}{15}\pi$$

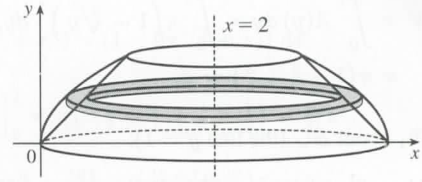




16. $y = \sqrt{x} \;\Rightarrow\; x = y^2$, so the outer radius is $2 - y^2$.

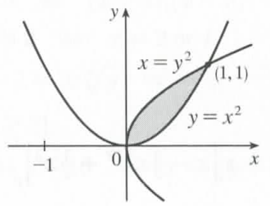
$$\begin{aligned} V &= \int_0^1 \pi\left[(2-y^2)^2 - (2-y)^2\right] dy \\ &= \pi \int_0^1 \left[(4 - 4y^2 + y^4) - (4 - 4y + y^2)\right] dy \\ &= \pi \int_0^1 (y^4 - 5y^2 + 4y)\, dy \\ &= \pi\left[\tfrac{1}{5}y^5 - \tfrac{5}{3}y^3 + 2y^2\right]_0^1 \\ &= \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right) = \tfrac{8}{15}\pi \end{aligned}$$

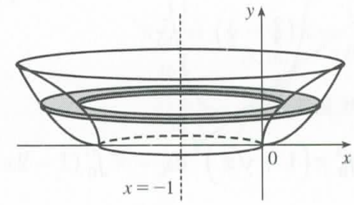




17. $y = x^2 \;\Rightarrow\; x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

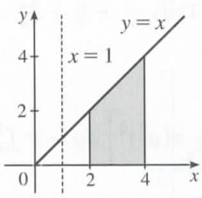
$$\begin{aligned} V &= \int_0^1 \pi\left\{\left[\sqrt{y} - (-1)\right]^2 - \left[y^2 - (-1)\right]^2\right\} dy = \pi \int_0^1 \left[\left(\sqrt{y} + 1\right)^2 - (y^2+1)^2\right] dy \\ &= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1\right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2\right) dy \\ &= \pi\left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{2}{3}y^3\right]_0^1 = \pi\left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3}\right) = \tfrac{29}{30}\pi \end{aligned}$$

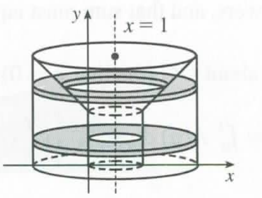




18. For $0 \leq y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$, the area of which is $A_1(y) = \pi(4-1)^2 - \pi(2-1)^2$. For $2 \leq y \leq 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$, the area of which is $A_2(y) = \pi(4-1)^2 - \pi(y-1)^2$.

$$\begin{aligned} V &= \int_0^4 A(y)\, dy = \pi \int_0^2 \left[(4-1)^2 - (2-1)^2\right] dy + \pi \int_2^4 \left[(4-1)^2 - (y-1)^2\right] dy \\ &= \pi\left[8y\right]_0^2 + \pi \int_2^4 (8 + 2y - y^2)\, dy \\ &= 16\pi + \pi\left[8y + y^2 - \tfrac{1}{3}y^3\right]_2^4 \\ &= 16\pi + \pi\left[\left(32 + 16 - \tfrac{64}{3}\right) - \left(16 + 4 - \tfrac{8}{3}\right)\right] \\ &= \tfrac{76}{3}\pi \end{aligned}$$





19. $\mathcal{R}_1$ about OA (the line $y = 0$): $V = \displaystyle\int_0^1 A(x)\, dx = \int_0^1 \pi(x^3)^2\, dx = \pi \int_0^1 x^6\, dx = \pi\left[\frac{1}{7}x^7\right]_0^1 = \frac{\pi}{7}$

20. $\mathcal{R}_1$ about OC (the line $x = 0$):

$$V = \int_0^1 A(y)\, dy = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt[3]{y}\right)^2\right] dy = \pi \int_0^1 (1 - y^{2/3})\, dy = \pi\left[y - \tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1 - \tfrac{3}{5}\right) = \tfrac{2}{5}\pi$$

21. $\mathcal{R}_1$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(1 - \sqrt[3]{y}\right)^2 dy = \pi \int_0^1 (1 - 2y^{1/3} + y^{2/3})\,dy = \pi\left[y - \tfrac{3}{2}y^{4/3} + \tfrac{3}{5}y^{5/3}\right]_0^1$$
$$= \pi\left(1 - \tfrac{3}{2} + \tfrac{3}{5}\right) = \tfrac{\pi}{10}$$

22. $\mathcal{R}_1$ about BC (the line $y = 1$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi(1 - x^3)^2\right] dx = \pi \int_0^1 \left[1 - (1 - 2x^3 + x^6)\right] dx$$
$$= \pi \int_0^1 (2x^3 - x^6)\,dx = \pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{7}\right) = \tfrac{5}{14}\pi$$

23. $\mathcal{R}_2$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt{x}\right)^2\right] dx = \pi \int_0^1 (1 - x)\,dx = \pi\left[x - \tfrac{1}{2}x^2\right]_0^1 = \pi\left(1 - \tfrac{1}{2}\right) = \tfrac{\pi}{2}$$

24. $\mathcal{R}_2$ about OC (the line $x = 0$): $V = \int_0^1 A(y)\,dy = \int_0^1 \pi(y^2)^2\,dy = \pi \int_0^1 y^4\,dy = \pi\left[\tfrac{1}{5}y^5\right]_0^1 = \tfrac{\pi}{5}$

25. $\mathcal{R}_2$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi(1 - y^2)^2\right] dy = \pi \int_0^1 \left[1 - (1 - 2y^2 + y^4)\right] dy = \pi \int_0^1 (2y^2 - y^4)\,dy$$
$$= \pi\left[\tfrac{2}{3}y^3 - \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{2}{3} - \tfrac{1}{5}\right) = \tfrac{7}{15}\pi$$

26. $\mathcal{R}_2$ about BC (the line $y = 1$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(1 - \sqrt{x}\right)^2 dx = \pi \int_0^1 (1 - 2x^{1/2} + x)\,dx = \pi\left[x - \tfrac{4}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^1 = \pi\left(1 - \tfrac{4}{3} + \tfrac{1}{2}\right) = \tfrac{\pi}{6}$$

27. $\mathcal{R}_3$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi\left(\sqrt{x}\right)^2 - \pi(x^3)^2\right] dx = \pi \int_0^1 (x - x^6)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{7}\right) = \tfrac{5}{14}\pi.$$

Note: Let $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$. If we rotate $\mathcal{R}$ about any of the segments OA, OC, AB, or BC, we obtain a right circular cylinder of height 1 and radius 1. Its volume is $\pi r^2 h = \pi(1)^2 \cdot 1 = \pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π. Thus, $\frac{\pi}{7} + \frac{\pi}{2} + \frac{5\pi}{14} = \left(\frac{2+7+5}{14}\right)\pi = \pi$.

28. $\mathcal{R}_3$ about OC (the line $x = 0$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi\left(\sqrt[3]{y}\right)^2 - \pi(y^2)^2\right] dy = \pi \int_0^1 (y^{2/3} - y^4)\,dy = \pi\left[\tfrac{3}{5}y^{5/3} - \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{3}{5} - \tfrac{1}{5}\right) = \tfrac{2}{5}\pi$$

Note: See the note in Exercise 27. For Exercises 20, 24, and 28, we have $\frac{2\pi}{5} + \frac{\pi}{5} + \frac{2\pi}{5} = \pi$.

29. $\mathcal{R}_3$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1 - y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy = \pi \int_0^1 \left[(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})\right] dy$$
$$= \pi \int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3})\,dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13}{30}\pi$$

Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have $\frac{\pi}{10} + \frac{7\pi}{15} + \frac{13\pi}{30} = \left(\frac{3+14+13}{30}\right)\pi = \pi$.

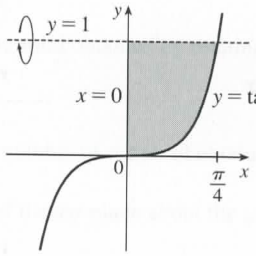
30. $\mathcal{R}_3$ about BC (the line $y = 1$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1-x^3)^2 - \pi\left(1-\sqrt{x}\right)^2\right]dx = \pi\int_0^1 \left[(1-2x^3+x^6)-(1-2x^{1/2}+x)\right]dx$$

$$= \pi\int_0^1 (-2x^3+x^6+2x^{1/2}-x)\,dx = \pi\left[-\tfrac{1}{2}x^4+\tfrac{1}{7}x^7+\tfrac{4}{3}x^{3/2}-\tfrac{1}{2}x^2\right]_0^1 = \pi\left(-\tfrac{1}{2}+\tfrac{1}{7}+\tfrac{4}{3}-\tfrac{1}{2}\right) = \tfrac{10}{21}\pi$$

Note: See the note in Exercise 27. For Exercises 22, 26, and 30, we have $\frac{5\pi}{14}+\frac{\pi}{6}+\frac{10\pi}{21} = \left(\frac{15+7+20}{42}\right)\pi = \pi$.

31. $V = \pi\displaystyle\int_0^{\pi/4} (1-\tan^3 x)^2\,dx$



32. $y = (x-2)^4$ and $8x - y = 16$ intersect when

$(x-2)^4 = 8x - 16 = 8(x-2) \quad\Leftrightarrow$

$(x-2)^4 - 8(x-2) = 0 \quad\Leftrightarrow\quad (x-2)[(x-2)^3 - 8] = 0 \quad\Leftrightarrow$

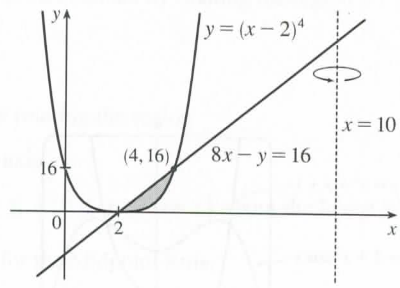
$x - 2 = 0$ or $x - 2 = 2 \quad\Leftrightarrow\quad x = 2$ or 4.

$y = (x-2)^4 \quad\Rightarrow\quad x - 2 = \pm\sqrt[4]{y} \quad\Rightarrow$

$x = 2 + \sqrt[4]{y}$ [since $x \geq 2$].

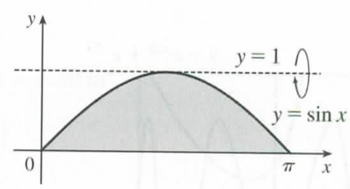
$8x - y = 16 \quad\Rightarrow\quad 8x = y + 16 \quad\Rightarrow\quad x = \frac{1}{8}y + 2$.

$$V = \pi\int_0^{16}\left\{\left[10-\left(\tfrac{1}{8}y+2\right)\right]^2 - \left[10-\left(2+\sqrt[4]{y}\right)\right]^2\right\}dy$$



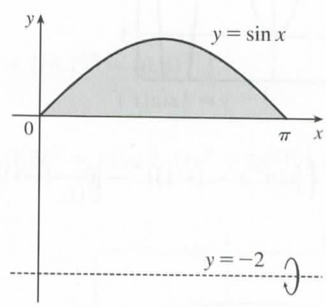
33. $V = \pi\displaystyle\int_0^{\pi}\left[(1-0)^2 - (1-\sin x)^2\right]dx$

$= \pi\displaystyle\int_0^{\pi}\left[1^2 - (1-\sin x)^2\right]dx$



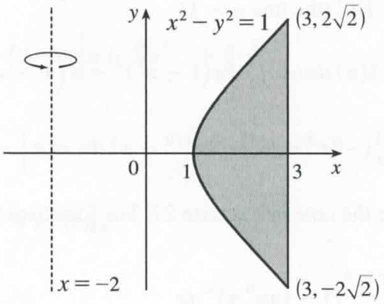
34. $V = \pi\displaystyle\int_0^{\pi}\left\{[\sin x - (-2)]^2 - [0-(-2)]^2\right\}dx$

$= \pi\displaystyle\int_0^{\pi}\left[(\sin x + 2)^2 - 2^2\right]dx$



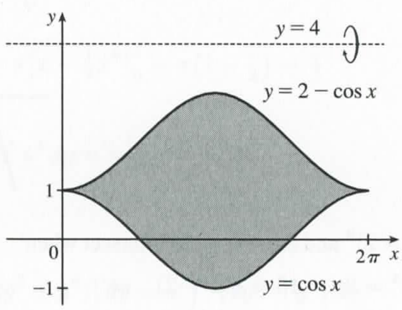
35. $V = \pi \int_{-\sqrt{8}}^{\sqrt{8}} \left\{ [3-(-2)]^2 - \left[\sqrt{y^2+1}-(-2)\right]^2 \right\} dy$

$= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[5^2 - \left(\sqrt{1+y^2}+2\right)^2 \right] dy$

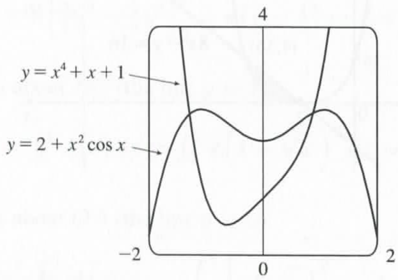


36. $V = \pi \int_0^{2\pi} \{(4-\cos x)^2 - [4-(2-\cos x)]^2\} \, dx$

$= \pi \int_0^{2\pi} \left[(4-\cos x)^2 - (2+\cos x)^2\right] dx$



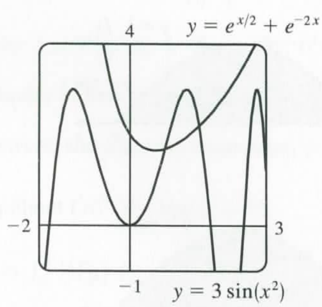
37.



$y = 2 + x^2 \cos x$ and $y = x^4 + x + 1$ intersect at $x = a \approx -1.288$ and $x = b \approx 0.884$.

$V = \pi \int_a^b [(2+x^2\cos x)^2 - (x^4+x+1)^2] dx \approx 23.780$

38.

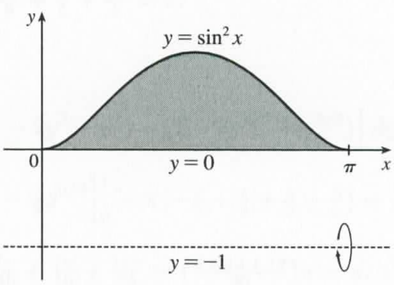


$y = 3\sin(x^2)$ and $y = e^{x/2} + e^{-2x}$ intersect at $x = a \approx 0.772$ and at $x = b \approx 1.524$.

$V = \pi \int_a^b \left\{ [3\sin(x^2)]^2 - (e^{x/2}+e^{-2x})^2 \right\} dx \approx 7.519$

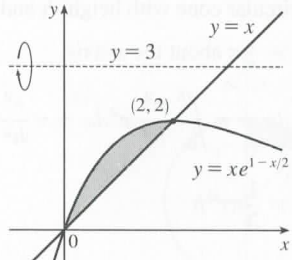
39. $V = \pi \int_0^{\pi} \left\{ \left[\sin^2 x - (-1)\right]^2 - [0-(-1)]^2 \right\} dx$

$\overset{\text{CAS}}{=} \frac{11}{8}\pi^2$



40. $V = \pi \int_0^2 \left[(3-x)^2 - (3 - xe^{1-x/2})^2 \right] dx$

$\overset{\text{CAS}}{=} \pi(-2e^2 + 24e - \frac{142}{3})$



41. $\pi \int_0^{\pi/2} \cos^2 x\, dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ of the xy-plane about the x-axis.

42. $\pi \int_2^5 y\, dy = \pi \int_2^5 \left(\sqrt{y}\right)^2 dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 2 \le y \le 5, 0 \le x \le \sqrt{y}\right\}$ of the xy-plane about the y-axis.

43. $\pi \int_0^1 (y^4 - y^8)\, dy = \pi \int_0^1 [(y^2)^2 - (y^4)^2]\, dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le y \le 1, y^4 \le x \le y^2\right\}$ of the xy-plane about the y-axis.

44. $\pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2]\, dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 1 \le y \le 1 + \cos x\right\}$ of the xy-plane about the x-axis.
Or: The solid could be obtained by rotating the region $\mathcal{R}' = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the line $y = -1$.

45. There are 10 subintervals over the 15-cm length, so we'll use $n = 10/2 = 5$ for the Midpoint Rule.

$V = \int_0^{15} A(x)\, dx \approx M_5 = \frac{15-0}{5}[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)]$

$= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3$

46. $V = \int_0^{10} A(x)\, dx \approx M_5 = \frac{10-0}{5}[A(1) + A(3) + A(5) + A(7) + A(9)]$

$= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 2(2.90) = 5.80 \text{ m}^3$

47. (a) $V = \int_2^{10} \pi [f(x)]^2\, dx \approx \pi \frac{10-2}{4} \left\{[f(3)]^2 + [f(5)]^2 + [f(7)]^2 + [f(9)]^2\right\}$

$\approx 2\pi \left[(1.5)^2 + (2.2)^2 + (3.8)^2 + (3.1)^2\right] \approx 196 \text{ units}^3$

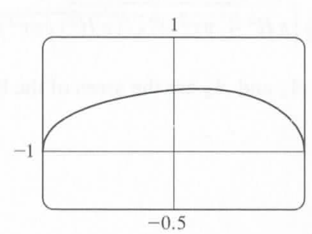
(b) $V = \int_0^4 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dy$

$\approx \pi \frac{4-0}{4} \left\{\left[(9.9)^2 - (2.2)^2\right] + \left[(9.7)^2 - (3.0)^2\right] + \left[(9.3)^2 - (5.6)^2\right] + \left[(8.7)^2 - (6.5)^2\right]\right\}$

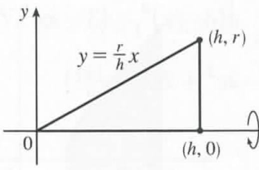
$\approx 838 \text{ units}^3$

48. (a) $V = \int_{-1}^1 \pi \left[\left(ax^3 + bx^2 + cx + d\right) \sqrt{1 - x^2}\right]^2 dx \overset{\text{CAS}}{=} \dfrac{4\left\{5a^2 + 18ac + 3\left[3b^2 + 14bd + 7(c^2 + 5d^2)\right]\right\}\pi}{315}$

(b) $y = (-0.06x^3 + 0.04x^2 + 0.1x + 0.54)\sqrt{1 - x^2}$ is graphed in the figure. Substitute $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$ in the answer for part (a) to get $V \overset{\text{CAS}}{=} \dfrac{3769\pi}{9375} \approx 1.263$.



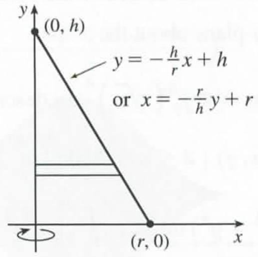
49. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2\, dx = \pi \frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\dfrac{r}{h}y + r$ about the y-axis.



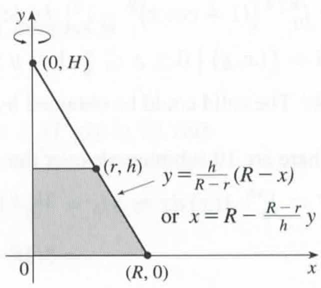
$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \overset{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$

$$= \pi \left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2 h - r^2 h + r^2 h\right) = \tfrac{1}{3}\pi r^2 h$$

* Or use substitution with $u = r - \dfrac{r}{h}y$ and $du = -\dfrac{r}{h}\,dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

50. $$V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy$$



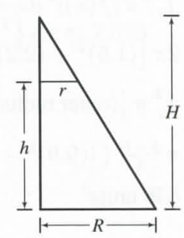
$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$$

$$= \pi \left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$$

$$= \pi\left[R^2 h - R(R-r)h + \tfrac{1}{3}(R-r)^2 h\right]$$

$$= \tfrac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

Another solution: $\dfrac{H}{R} = \dfrac{H-h}{r}$ by similar triangles. Therefore,



$Hr = HR - hR \;\Rightarrow\; hR = H(R-r) \;\Rightarrow\; H = \dfrac{hR}{R-r}$. Now

$V = \tfrac{1}{3}\pi R^2 H - \tfrac{1}{3}\pi r^2 (H-h)$ [by Exercise 49]

$$= \tfrac{1}{3}\pi R^2 \frac{hR}{R-r} - \tfrac{1}{3}\pi r^2 \frac{rh}{R-r} \qquad \left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)}\right]$$

$$= \frac{1}{3}\pi h \frac{R^3 - r^3}{R-r} = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

$$= \tfrac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]h = \tfrac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$$

where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 52 for a related result.)

51. $x^2 + y^2 = r^2 \iff x^2 = r^2 - y^2$

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy = \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r}$$

$$= \pi\left\{\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right\}$$

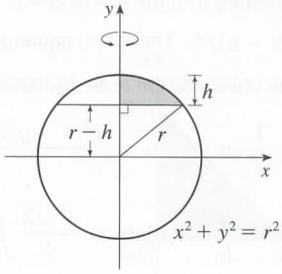
$$= \pi\{\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)[3r^2 - (r-h)^2]\}$$

$$= \tfrac{1}{3}\pi\{2r^3 - (r-h)[3r^2 - (r^2 - 2rh + h^2)]\}$$

$$= \tfrac{1}{3}\pi\{2r^3 - (r-h)[2r^2 + 2rh - h^2]\}$$

$$= \tfrac{1}{3}\pi(2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3)$$

$$= \tfrac{1}{3}\pi(3rh^2 - h^3) = \tfrac{1}{3}\pi h^2(3r - h), \text{ or, equivalently, } \pi h^2\left(r - \frac{h}{3}\right)$$



52. An equation of the line is $x = \dfrac{\Delta x}{\Delta y}y + (x\text{-intercept}) = \dfrac{a/2 - b/2}{h - 0}y + \dfrac{b}{2} = \dfrac{a-b}{2h}y + \dfrac{b}{2}$.

$$V = \int_0^h A(y)\,dy = \int_0^h (2x)^2\,dy$$

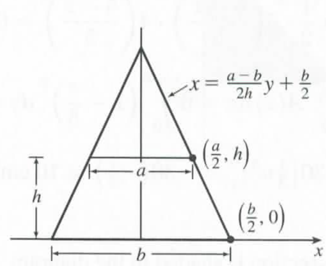
$$= \int_0^h \left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right]^2 dy = \int_0^h \left[\frac{a-b}{h}y + b\right]^2 dy$$

$$= \int_0^h \left[\frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2\right] dy$$

$$= \left[\frac{(a-b)^2}{3h^2}y^3 + \frac{b(a-b)}{h}y^2 + b^2 y\right]_0^h$$

$$= \tfrac{1}{3}(a-b)^2h + b(a-b)h + b^2h = \tfrac{1}{3}(a^2 - 2ab + b^2 + 3ab)h$$

$$= \tfrac{1}{3}(a^2 + ab + b^2)h$$



[Note that this can be written as $\frac{1}{3}(A_1 + A_2 + \sqrt{A_1 A_2})h$, as in Exercise 50.]

If $a = b$, we get a rectangular solid with volume b^2h. If $a = 0$, we get a square pyramid with volume $\frac{1}{3}b^2h$.

53. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \frac{y}{h}\right)$.

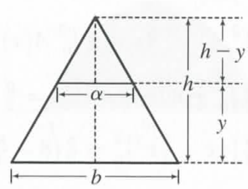
Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \dfrac{y}{h}\right)$. So

$$V = \int_0^h A(y)\,dy = \int_0^h \left[b\left(1 - \tfrac{y}{h}\right)\right]\left[2b\left(1 - \tfrac{y}{h}\right)\right] dy$$

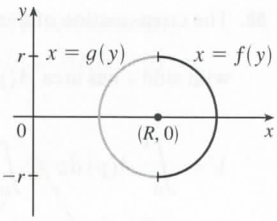
$$= \int_0^h 2b^2\left(1 - \tfrac{y}{h}\right)^2 dy = 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy$$

$$= 2b^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = 2b^2\left[h - h + \tfrac{1}{3}h\right]$$

$$= \tfrac{2}{3}b^2h \quad [= \tfrac{1}{3}Bh \text{ where } B \text{ is the area of the base, as with any pyramid.}]$$



63. (a) The torus is obtained by rotating the circle $(x - R)^2 + y^2 = r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x = R + \sqrt{r^2 - y^2} = f(y)$ and the left half by $x = R - \sqrt{r^2 - y^2} = g(y)$. So



$$\begin{aligned} V &= \pi \textstyle\int_{-r}^{r} \{[f(y)]^2 - [g(y)]^2\}\, dy \\ &= 2\pi \textstyle\int_0^r \left[\left(R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)\right] dy \\ &= 2\pi \textstyle\int_0^r 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy \end{aligned}$$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so

$8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy = 8\pi R \cdot \frac{1}{4}\pi r^2 = 2\pi^2 r^2 R$.

64. The cross-sections perpendicular to the y-axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16 - y^2}$ in the xy-plane and a height of $\frac{1}{\sqrt{3}}\, y$, since $\angle BAC = 30°$ and $|BC| = \frac{1}{\sqrt{3}}\, |AB|$. Thus, $A(y) = \frac{2}{\sqrt{3}}\, y\sqrt{16 - y^2}$ and

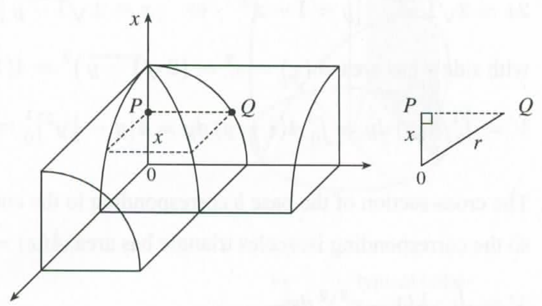
$$V = \textstyle\int_0^4 A(y)\, dy = \frac{2}{\sqrt{3}} \int_0^4 \sqrt{16 - y^2}\, y\, dy = \frac{2}{\sqrt{3}} \int_{16}^0 u^{1/2}\left(-\frac{1}{2}\, du\right) \qquad [\text{Put } u = 16 - y^2, \text{ so } du = -2y\, dy]$$

$$= \frac{1}{\sqrt{3}} \int_0^{16} u^{1/2}\, du = \frac{1}{\sqrt{3}} \frac{2}{3}\left[u^{3/2}\right]_0^{16} = \frac{2}{3\sqrt{3}}\,(64) = \frac{128}{3\sqrt{3}}$$

65. (a) Volume$(S_1) = \int_0^h A(z)\, dz =$ Volume(S_2) since the cross-sectional area $A(z)$ at height z is the same for both solids.

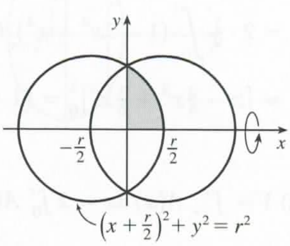
(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h, that is, $\pi r^2 h$.

66. Each cross-section of the solid S in a plane perpendicular to the x-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2 = r^2 - x^2$. Therefore, $A(x) = 4(r^2 - x^2)$ and the volume of S is



$$\begin{aligned} V &= \textstyle\int_{-r}^{r} A(x)\, dx = 4\int_{-r}^{r} (r^2 - x^2)\, dx \\ &= 8(r^2 - x^2)\, dx = 8\left[r^2 x - \frac{1}{3}x^3\right]_0^r = \frac{16}{3} r^3 \end{aligned}$$

67. The volume is obtained by rotating the area common to two circles of radius r, as shown. The volume of the right half is



$$V_{\text{right}} = \pi \int_0^{r/2} y^2\, dx = \pi \int_0^{r/2} \left[r^2 - \left(\tfrac{1}{2}r + x\right)^2\right] dx$$

$$= \pi\left[r^2 x - \tfrac{1}{3}\left(\tfrac{1}{2}r + x\right)^3\right]_0^{r/2} = \pi\left[\left(\tfrac{1}{2}r^3 - \tfrac{1}{3}r^3\right) - \left(0 - \tfrac{1}{24}r^3\right)\right] = \tfrac{5}{24}\pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 51 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi h^2(3r - h) = \frac{2}{3}\pi\left(\frac{1}{2}r\right)^2\left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.