6 □ APPLICATIONS OF INTEGRATION

6.1 Areas Between Curves

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

2. $A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx = \left[\frac{2}{3}(x+2)^{3/2} - \ln(x+1)\right]_0^2$

$= \left[\frac{2}{3}(4)^{3/2} - \ln 3\right] - \left[\frac{2}{3}(2)^{3/2} - \ln 1\right] = \frac{16}{3} - \ln 3 - \frac{4}{3}\sqrt{2}$

3. $A = \int_{y=-1}^{y=1} (x_R - x_L)\,dy = \int_{-1}^1 \left[e^y - (y^2 - 2)\right] dy = \int_{-1}^1 (e^y - y^2 + 2)\,dy$

$= \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^1 = \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - \frac{1}{e} + \frac{10}{3}$

4. $A = \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy = \int_0^3 (-2y^2 + 6y)\,dy = \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = (-18 + 27) - 0 = 9$

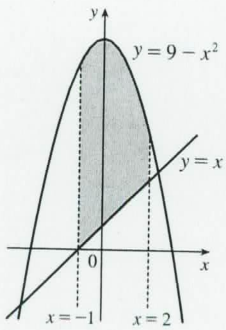
5. $A = \int_{-1}^2 \left[(9 - x^2) - (x+1)\right] dx$

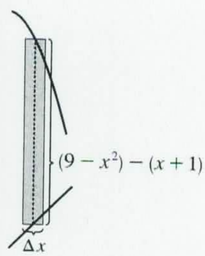
$= \int_{-1}^2 (8 - x - x^2)\,dx$

$= \left[8x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$

$= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right)$

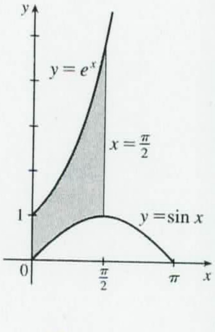
$= 22 - 3 + \frac{1}{2} = \frac{39}{2}$

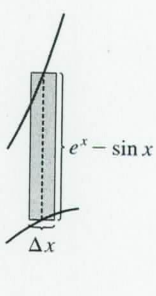




6. $A = \int_0^{\pi/2} (e^x - \sin x)\,dx = \left[e^x + \cos x\right]_0^{\pi/2}$

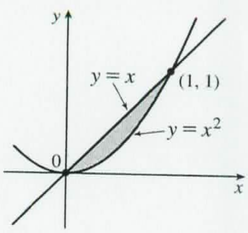
$= (e^{\pi/2} + 0) - (1 + 1) = e^{\pi/2} - 2$

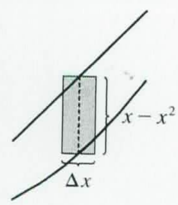




7. The curves intersect when $x = x^2 \Leftrightarrow x^2 - x = 0 \Leftrightarrow$ $x(x-1) = 0 \Leftrightarrow x = 0$ or 1.

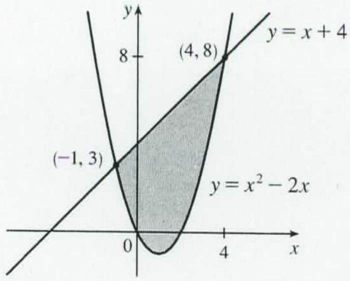
$A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

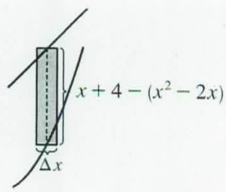




8. The curves intersect when $x^2 - 2x = x + 4 \quad \Leftrightarrow \quad x^2 - 3x - 4 = 0 \quad \Leftrightarrow \quad (x+1)(x-4) = 0 \quad \Leftrightarrow \quad x = -1$ or 4.

$$\begin{aligned} A &= \int_{-1}^{4} \left[x + 4 - (x^2 - 2x)\right] dx \\ &= \int_{-1}^{4} (-x^2 + 3x + 4)\, dx \\ &= \left[-\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 4x\right]_{-1}^{4} \\ &= \left(-\tfrac{64}{3} + 24 + 16\right) - \left(\tfrac{1}{3} + \tfrac{3}{2} - 4\right) \\ &= \tfrac{125}{6} \end{aligned}$$

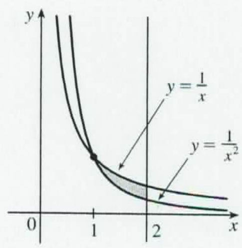


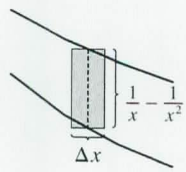


9. $A = \displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \left[\ln x + \frac{1}{x}\right]_1^2$

$= \left(\ln 2 + \tfrac{1}{2}\right) - (\ln 1 + 1)$

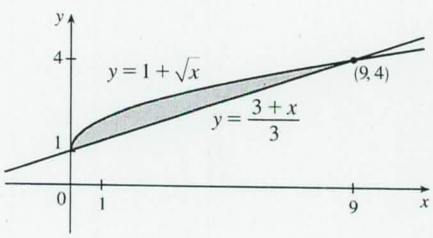
$= \ln 2 - \tfrac{1}{2} \approx 0.19$

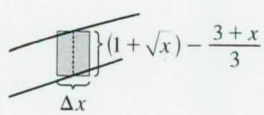




10. $1 + \sqrt{x} = \dfrac{3 + x}{3} = 1 + \dfrac{x}{3} \quad \Rightarrow \quad \sqrt{x} = \dfrac{x}{3} \quad \Rightarrow \quad x = \dfrac{x^2}{9} \quad \Rightarrow \quad 9x - x^2 = 0 \quad \Rightarrow \quad x(9 - x) = 0 \quad \Rightarrow \quad x = 0$ or 9, so

$$\begin{aligned} A &= \int_0^9 \left[(1 + \sqrt{x}) - \left(\frac{3 + x}{3}\right)\right] dx = \int_0^9 \left[(1 + \sqrt{x}) - \left(1 + \frac{x}{3}\right)\right] dx \\ &= \int_0^9 \left(\sqrt{x} - \tfrac{1}{3}x\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{6}x^2\right]_0^9 = 18 - \tfrac{27}{2} = \tfrac{9}{2} \end{aligned}$$

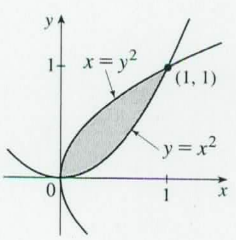


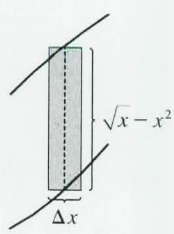


11. $A = \displaystyle\int_0^1 \left(\sqrt{x} - x^2\right) dx$

$= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1$

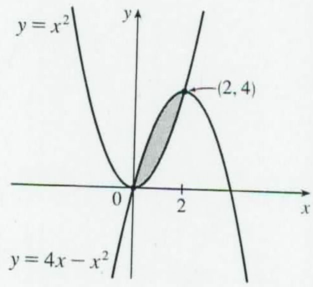
$= \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3}$

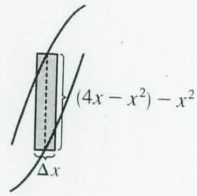




12. $x^2 = 4x - x^2 \quad \Leftrightarrow \quad 2x^2 - 4x = 0 \quad \Leftrightarrow \quad 2x(x-2) = 0 \quad \Leftrightarrow \quad x = 0$ or 2, so

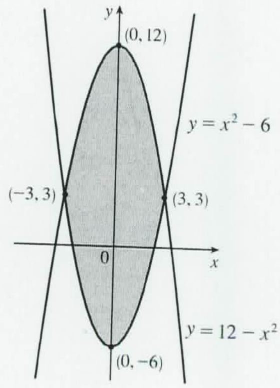
$$\begin{aligned} A &= \int_0^2 \left[(4x - x^2) - x^2\right]\,dx \\ &= \int_0^2 (4x - 2x^2)\,dx \\ &= \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2 \\ &= 8 - \tfrac{16}{3} = \tfrac{8}{3} \end{aligned}$$

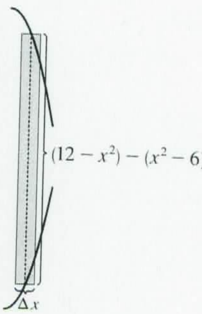




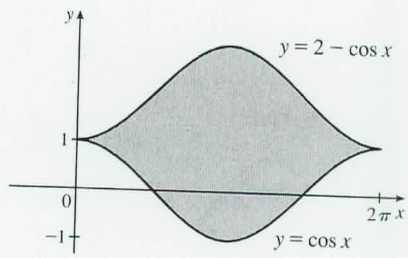
13. $12 - x^2 = x^2 - 6 \quad \Leftrightarrow \quad 2x^2 = 18 \quad \Leftrightarrow$
$x^2 = 9 \quad \Leftrightarrow \quad x = \pm 3$, so

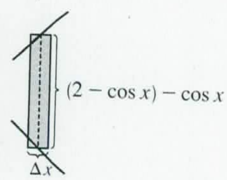
$$\begin{aligned} A &= \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6)\right]\,dx \\ &= 2\int_0^3 (18 - 2x^2)\,dx \qquad \text{[by symmetry]} \\ &= 2\left[18x - \tfrac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right] \\ &= 2(36) = 72 \end{aligned}$$





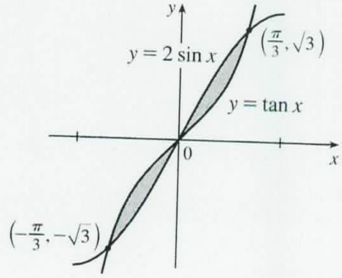
14. $$\begin{aligned} A &= \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right]\,dx \\ &= \int_0^{2\pi} (2 - 2\cos x)\,dx \\ &= \Big[2x - 2\sin x\Big]_0^{2\pi} \\ &= (4\pi - 0) - 0 = 4\pi \end{aligned}$$

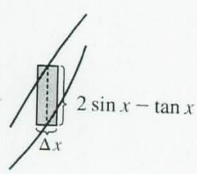




15. The curves intersect when $\tan x = 2\sin x$ (on $[-\pi/3, \pi/3]$) $\Leftrightarrow \quad \sin x = 2\sin x\,\cos x \quad \Leftrightarrow$
$2\sin x\,\cos x - \sin x = 0 \quad \Leftrightarrow \quad \sin x\,(2\cos x - 1) = 0 \quad \Leftrightarrow \quad \sin x = 0$ or $\cos x = \frac{1}{2} \quad \Leftrightarrow \quad x = 0$ or $x = \pm\frac{\pi}{3}$.

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} (2\sin x - \tan x)\,dx \\ &= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx \qquad \text{[by symmetry]} \\ &= 2\Big[-2\cos x - \ln|\sec x|\Big]_0^{\pi/3} \\ &= 2\left[(-1 - \ln 2) - (-2 - 0)\right] \\ &= 2(1 - \ln 2) = 2 - 2\ln 2 \end{aligned}$$





16. $x^3 - x = 3x \iff x^3 - 4x = 0 \iff$

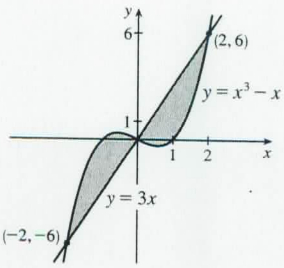
$x(x^2 - 4) = 0 \iff x(x+2)(x-2) = 0 \iff$

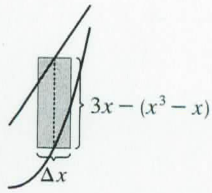
$x = 0, -2$, or 2.

$$A = \int_{-2}^{2} \left|3x - (x^3 - x)\right| dx$$

$$= 2\int_0^2 \left[3x - (x^3 - x)\right] dx \qquad \text{[by symmetry]}$$

$$= 2\int_0^2 (4x - x^3)\, dx = 2\left[2x^2 - \tfrac{1}{4}x^4\right]_0^2 = 2(8-4) = 8$$

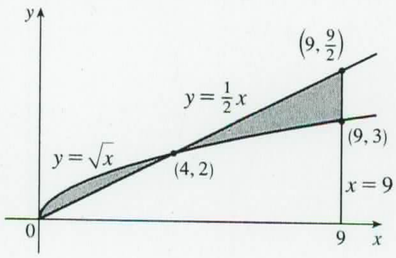


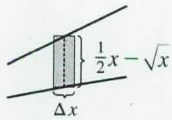


17. $\frac{1}{2}x = \sqrt{x} \;\Rightarrow\; \frac{1}{4}x^2 = x \;\Rightarrow\; x^2 - 4x = 0 \;\Rightarrow\; x(x-4) = 0 \;\Rightarrow\; x = 0$ or 4, so

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$

$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$





For $4 < x < 9$

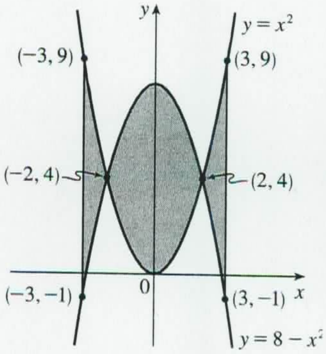
18. $A = \int_{-3}^{3} \left|(8 - x^2) - x^2\right| dx = 2\int_0^3 \left|8 - 2x^2\right| dx$

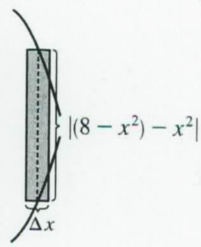
$$= 2\int_0^2 (8 - 2x^2)\, dx + 2\int_2^3 (2x^2 - 8)\, dx$$

$$= 2\left[8x - \tfrac{2}{3}x^3\right]_0^2 + 2\left[\tfrac{2}{3}x^3 - 8x\right]_2^3$$

$$= 2\left[\left(16 - \tfrac{16}{3}\right) - 0\right] + 2\left[(18 - 24) - \left(\tfrac{16}{3} - 16\right)\right]$$

$$= 32 - \tfrac{32}{3} + 20 - \tfrac{32}{3} = 52 - \tfrac{64}{3} = \tfrac{92}{3}$$



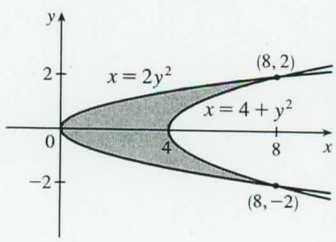


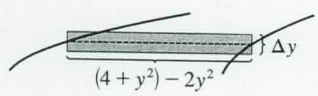
19. $2y^2 = 4 + y^2 \iff y^2 = 4 \iff y = \pm 2$, so

$$A = \int_{-2}^{2} \left[(4 + y^2) - 2y^2\right] dy$$

$$= 2\int_0^2 (4 - y^2)\, dy \qquad \text{[by symmetry]}$$

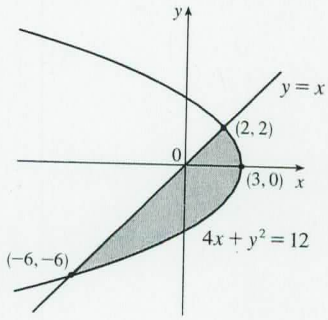
$$= 2\left[4y - \tfrac{1}{3}y^3\right]_0^2 = 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3}$$

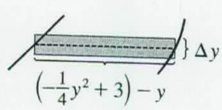




20. $4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow \quad x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

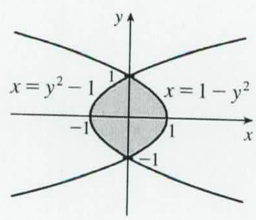
$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy = \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2} = \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18) = 22 - \tfrac{2}{3} = \tfrac{64}{3}.$$

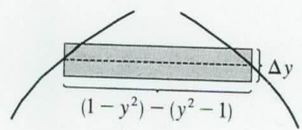




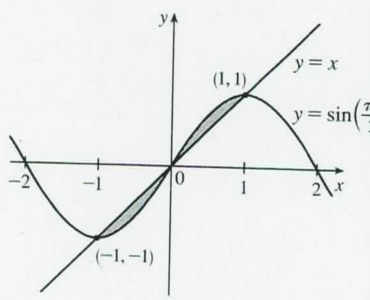
21. The curves intersect when $1 - y^2 = y^2 - 1 \quad \Leftrightarrow \quad 2 = 2y^2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm 1.$

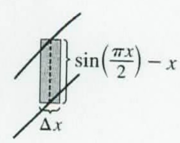
$$\begin{aligned} A &= \int_{-1}^{1} \left[(1 - y^2) - (y^2 - 1)\right] dy \\ &= \int_{-1}^{1} 2(1 - y^2)\, dy \\ &= 2 \cdot 2 \int_{0}^{1} (1 - y^2)\, dy \\ &= 4\left[y - \tfrac{1}{3}y^3\right]_0^1 = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3} \end{aligned}$$





22.
$$\begin{aligned} A &= 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx \\ &= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 \\ &= 2\left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right)\right] \\ &= \frac{4}{\pi} - 1 \end{aligned}$$



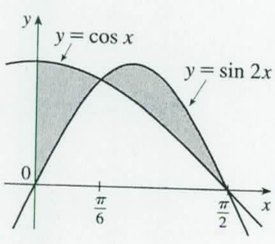


23. Notice that $\cos x = \sin 2x = 2\sin x \cos x \quad \Leftrightarrow$

$2\sin x \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$

$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

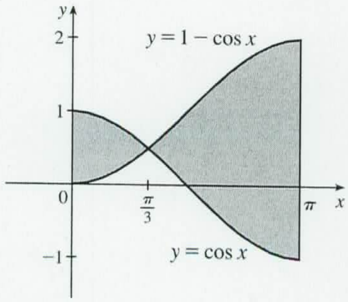
$$\begin{aligned} A &= \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx \\ &= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} - \left(0 + \tfrac{1}{2} \cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2} \cdot \tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2} \end{aligned}$$

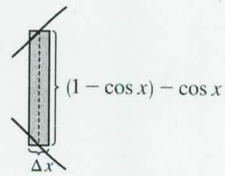


24. The curves intersect when $\cos x = 1 - \cos x$ (on $[0, \pi]$) $\Leftrightarrow$ $2\cos x = 1$ $\Leftrightarrow$ $\cos x = \frac{1}{2}$ $\Leftrightarrow$ $x = \frac{\pi}{3}$.

$$A = \int_0^{\pi/3} [\cos x - (1 - \cos x)]\,dx + \int_{\pi/3}^{\pi} [(1 - \cos x) - \cos x]\,dx = \int_0^{\pi/3} (2\cos x - 1)\,dx + \int_{\pi/3}^{\pi} (1 - 2\cos x)\,dx$$

$$= \Big[2\sin x - x\Big]_0^{\pi/3} + \Big[x - 2\sin x\Big]_{\pi/3}^{\pi} = \left(\sqrt{3} - \frac{\pi}{3}\right) - 0 + (\pi - 0) - \left(\frac{\pi}{3} - \sqrt{3}\right) = 2\sqrt{3} + \frac{\pi}{3}$$





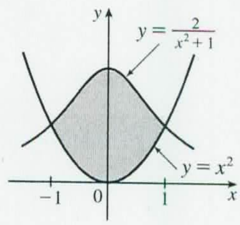
25. The curves intersect when $x^2 = \dfrac{2}{x^2+1}$ $\Leftrightarrow$

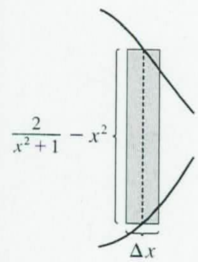
$x^4 + x^2 = 2$ $\Leftrightarrow$ $x^4 + x^2 - 2 = 0$ $\Leftrightarrow$

$(x^2+2)(x^2-1) = 0$ $\Leftrightarrow$ $x^2 = 1$ $\Leftrightarrow$ $x = \pm 1$.

$$A = \int_{-1}^{1} \left(\frac{2}{x^2+1} - x^2\right) dx = 2\int_0^1 \left(\frac{2}{x^2+1} - x^2\right) dx$$

$$= 2\Big[2\tan^{-1} x - \tfrac{1}{3}x^3\Big]_0^1 = 2\big(2 \cdot \tfrac{\pi}{4} - \tfrac{1}{3}\big) = \pi - \tfrac{2}{3} \approx 2.47$$





26. For $x > 0$, $x = x^2 - 2$ $\Rightarrow$ $0 = x^2 - x - 2$ $\Rightarrow$ $0 = (x-2)(x+1)$ $\Rightarrow$ $x = 2$. By symmetry,

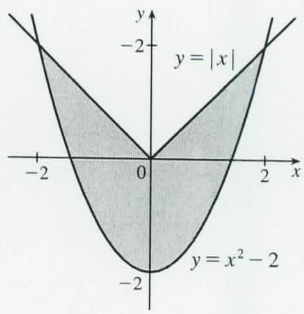
$$A = \int_{-2}^{2} [|x| - (x^2 - 2)]\,dx$$

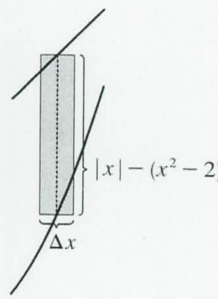
$$= 2\int_0^2 [x - (x^2 - 2)]\,dx$$

$$= 2\int_0^2 (x - x^2 + 2)\,dx$$

$$= 2\big[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x\big]_0^2$$

$$= 2\big(2 - \tfrac{8}{3} + 4\big) = \tfrac{20}{3}$$





27. $1/x = x$ $\Leftrightarrow$ $1 = x^2$ $\Leftrightarrow$ $x = \pm 1$ and $1/x = \frac{1}{4}x$ $\Leftrightarrow$

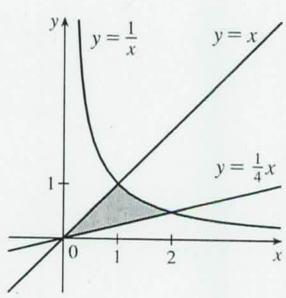
$4 = x^2$ $\Leftrightarrow$ $x = \pm 2$, so for $x > 0$,

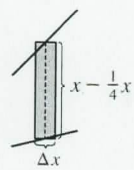
$$A = \int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

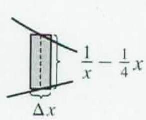
$$= \int_0^1 \left(\frac{3}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

$$= \big[\tfrac{3}{8}x^2\big]_0^1 + \big[\ln|x| - \tfrac{1}{8}x^2\big]_1^2$$

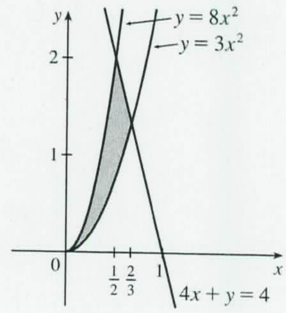
$$= \tfrac{3}{8} + \big(\ln 2 - \tfrac{1}{2}\big) - \big(0 - \tfrac{1}{8}\big) = \ln 2$$

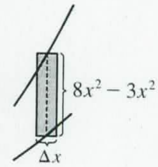


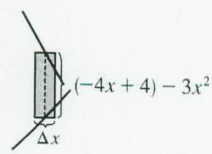




28. The curves $y = 3x^2$ and $y = -4x + 4$ intersect when $3x^2 = -4x + 4$ [for $x \geq 0$] $\Leftrightarrow$ $3x^2 + 4x - 4 = 0$ $\Leftrightarrow$ $(3x - 2)(x + 2) = 0$ $\Rightarrow$ $x = \frac{2}{3}$. The curves $y = 8x^2$ and $y = -4x + 4$ intersect when $8x^2 = -4x + 4$ [for $x \geq 0$] $\Leftrightarrow$ $8x^2 + 4x - 4 = 0$ $\Leftrightarrow$ $2x^2 + x - 1 = 0$ $\Leftrightarrow$ $(2x - 1)(x + 1) = 0$ $\Rightarrow$ $x = \frac{1}{2}$.







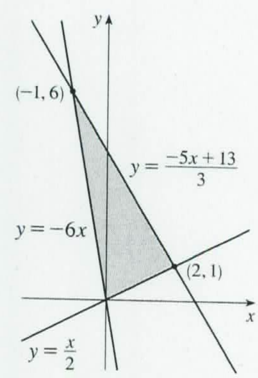
$$A = \int_0^{1/2} (8x^2 - 3x^2)\,dx + \int_{1/2}^{2/3} \left[(-4x + 4) - 3x^2\right] dx$$

$$= \int_0^{1/2} 5x^2\,dx + \int_{1/2}^{2/3} (-3x^2 - 4x + 4)\,dx = \left[\tfrac{5}{3}x^3\right]_0^{1/2} + \left[-x^3 - 2x^2 + 4x\right]_{1/2}^{2/3}$$

$$= \tfrac{5}{3}\left(\tfrac{1}{2}\right)^3 - 0 + \left[-\left(\tfrac{2}{3}\right)^3 - 2\left(\tfrac{2}{3}\right)^2 + 4\left(\tfrac{2}{3}\right)\right] - \left[-\left(\tfrac{1}{2}\right)^3 - 2\left(\tfrac{1}{2}\right)^2 + 4\left(\tfrac{1}{2}\right)\right] = \tfrac{5}{24} - \tfrac{8}{27} - \tfrac{8}{9} + \tfrac{8}{3} + \tfrac{1}{8} + \tfrac{1}{2} - 2$$

$$= \tfrac{45}{216} - \tfrac{64}{216} - \tfrac{192}{216} + \tfrac{576}{216} + \tfrac{27}{216} + \tfrac{108}{216} - \tfrac{432}{216} = \tfrac{68}{216} = \tfrac{17}{54} \quad [\approx 0.315]$$

29. An equation of the line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$; through $(0, 0)$ and $(-1, 6)$ is $y = -6x$; through $(2, 1)$ and $(-1, 6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.



$$A = \int_{-1}^{0} \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right] dx + \int_0^2 \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right] dx$$

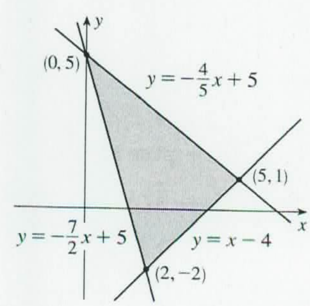
$$= \int_{-1}^{0} \left(\tfrac{13}{3}x + \tfrac{13}{3}\right) dx + \int_0^2 \left(-\tfrac{13}{6}x + \tfrac{13}{3}\right) dx$$

$$= \tfrac{13}{3} \int_{-1}^{0} (x + 1)\,dx + \tfrac{13}{3} \int_0^2 \left(-\tfrac{1}{2}x + 1\right) dx$$

$$= \tfrac{13}{3}\left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \tfrac{13}{3}\left[-\tfrac{1}{4}x^2 + x\right]_0^2$$

$$= \tfrac{13}{3}\left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}\left[(-1 + 2) - 0\right] = \tfrac{13}{3} \cdot \tfrac{1}{2} + \tfrac{13}{3} \cdot 1 = \tfrac{13}{2}$$

30. $A = \int_0^2 \left[\left(-\tfrac{4}{5}x + 5\right) - \left(-\tfrac{7}{2}x + 5\right)\right] dx + \int_2^5 \left[\left(-\tfrac{4}{5}x + 5\right) - (x - 4)\right] dx$



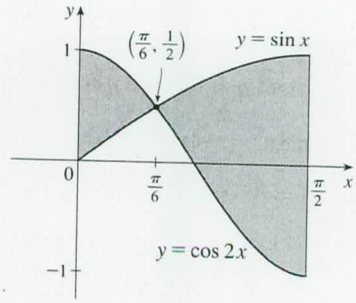
$$= \int_0^2 \tfrac{27}{10}x\,dx + \int_2^5 \left(-\tfrac{9}{5}x + 9\right) dx$$

$$= \left[\tfrac{27}{20}x^2\right]_0^2 + \left[-\tfrac{9}{10}x^2 + 9x\right]_2^5$$

$$= \left(\tfrac{27}{5} - 0\right) + \left(-\tfrac{45}{2} + 45\right) - \left(-\tfrac{18}{5} + 18\right) = \tfrac{27}{2}$$

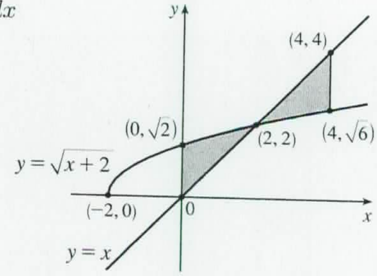
31. The curves intersect when $\sin x = \cos 2x$ (on $[0, \pi/2]$) $\Leftrightarrow$ $\sin x = 1 - 2\sin^2 x$ $\Leftrightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Leftrightarrow$ $(2\sin x - 1)(\sin x + 1) = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}$ $\Rightarrow$ $x = \frac{\pi}{6}$.

$$\begin{aligned} A &= \int_0^{\pi/2} |\sin x - \cos 2x|\, dx \\ &= \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x)\, dx \\ &= \left[\tfrac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \tfrac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} \\ &= \left(\tfrac{1}{4}\sqrt{3} + \tfrac{1}{2}\sqrt{3}\right) - (0+1) + (0-0) - \left(-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\sqrt{3}\right) \\ &= \tfrac{3}{2}\sqrt{3} - 1 \end{aligned}$$



32. The curves intersect when $\sqrt{x+2} = x$ $\Rightarrow$ $x + 2 = x^2$ $\Rightarrow$ $x^2 - x - 2 = 0$ $\Rightarrow$ $(x-2)(x+1) = 0$ $\Rightarrow$ $x = -1$ or 2. [-1 is extraneous]

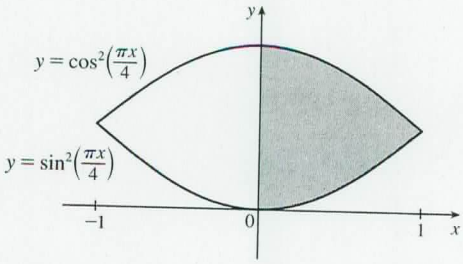
$$\begin{aligned} A &= \int_0^4 \left|\sqrt{x+2} - x\right| dx = \int_0^2 \left(\sqrt{x+2} - x\right) dx + \int_2^4 \left(x - \sqrt{x+2}\right) dx \\ &= \left[\tfrac{2}{3}(x+2)^{3/2} - \tfrac{1}{2}x^2\right]_0^2 + \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}(x+2)^{3/2}\right]_2^4 \\ &= \left(\tfrac{16}{3} - 2\right) - \left[\tfrac{2}{3}(2\sqrt{2}) - 0\right] + \left[8 - \tfrac{2}{3}(6\sqrt{6})\right] - \left(2 - \tfrac{16}{3}\right) \\ &= 4 + \tfrac{32}{3} - \tfrac{4}{3}\sqrt{2} - 4\sqrt{6} = \tfrac{44}{3} - 4\sqrt{6} - \tfrac{4}{3}\sqrt{2} \end{aligned}$$



33. Let $f(x) = \cos^2\left(\frac{\pi x}{4}\right) - \sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x = \frac{1-0}{4}$.

The shaded area is given by

$$\begin{aligned} A = \textstyle\int_0^1 f(x)\, dx &\approx M_4 \\ &= \tfrac{1}{4}\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right)\right] \\ &\approx 0.6407 \end{aligned}$$

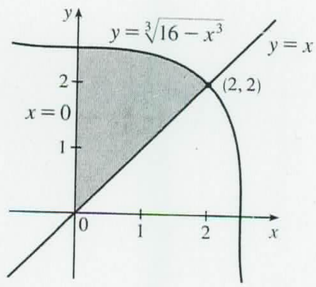


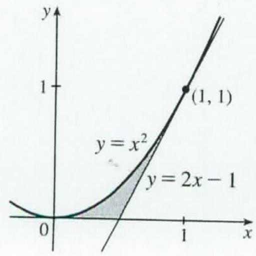
34. The curves intersect when $\sqrt[3]{16 - x^3} = x$ $\Rightarrow$ $16 - x^3 = x^3$ $\Rightarrow$ $2x^3 = 16$ $\Rightarrow$ $x^3 = 8$ $\Rightarrow$ $x = 2$.

Let $f(x) = \sqrt[3]{16 - x^3} - x$ and $\Delta x = \frac{2-0}{4}$.

The shaded area is given by

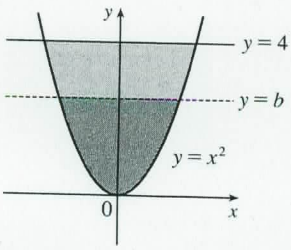
$$\begin{aligned} A = \textstyle\int_0^2 f(x)\, dx &\approx M_4 \\ &= \tfrac{2}{4}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right] \\ &\approx 2.8144 \end{aligned}$$



48. 

We start by finding the equation of the tangent line to $y = x^2$ at the point $(1, 1)$: $y' = 2x$, so the slope of the tangent is $2(1) = 2$, and its equation is $y - 1 = 2(x - 1)$, or $y = 2x - 1$. We would need two integrals to integrate with respect to x, but only one to integrate with respect to y.

$$A = \int_0^1 \left[\tfrac{1}{2}(y+1) - \sqrt{y}\,\right] dy = \left[\tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\right]_0^1$$
$$= \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3} = \tfrac{1}{12}$$

49. 

By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \;\Rightarrow\; x = \sqrt{y}$. We are looking for a number b such that

$$\int_0^b \sqrt{y}\,dy = \int_b^4 \sqrt{y}\,dy \;\Rightarrow\; \tfrac{2}{3}\left[y^{3/2}\right]_0^b = \tfrac{2}{3}\left[y^{3/2}\right]_b^4 \;\Rightarrow$$
$$b^{3/2} = 4^{3/2} - b^{3/2} \;\Rightarrow\; 2b^{3/2} = 8 \;\Rightarrow\; b^{3/2} = 4 \;\Rightarrow\; b = 4^{2/3} \approx 2.52.$$

50. (a) We want to choose a so that

$$\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \;\Rightarrow\; \left[\frac{-1}{x}\right]_1^a = \left[\frac{-1}{x}\right]_a^4 \;\Rightarrow\; -\frac{1}{a} + 1 = -\frac{1}{4} + \frac{1}{a} \;\Rightarrow\; \frac{5}{4} = \frac{2}{a} \;\Rightarrow\; a = \frac{8}{5}.$$

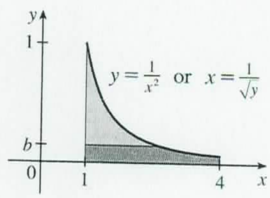
(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)]. Now the line $y = b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $x = 4$, since the area under the line $y = 1/4^2$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$. This implies that

$\int_b^1 (1/\sqrt{y} - 1)\,dy = \frac{1}{2}\cdot\frac{3}{4} \;\Rightarrow\; \left[2\sqrt{y} - y\right]_b^1 = \frac{3}{8} \;\Rightarrow\; 1 - 2\sqrt{b} + b = \frac{3}{8} \;\Rightarrow$

$b - 2\sqrt{b} + \frac{5}{8} = 0$. Letting $c = \sqrt{b}$, we get $c^2 - 2c + \frac{5}{8} = 0 \;\Rightarrow$

$8c^2 - 16c + 5 = 0$. Thus, $c = \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}$. But $c = \sqrt{b} < 1 \;\Rightarrow$

$c = 1 - \frac{\sqrt{6}}{4} \;\Rightarrow\; b = c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8}(11 - 4\sqrt{6}) \approx 0.1503.$



51. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

$A = 4\int_0^c (c^2 - x^2)\,dx = 4\left[c^2x - \frac{1}{3}x^3\right]_0^c = 4\left(c^3 - \frac{1}{3}c^3\right) = 4\left(\frac{2}{3}c^3\right) = \frac{8}{3}c^3$

So $A = 576 \;\Leftrightarrow\; \frac{8}{3}c^3 = 576 \;\Leftrightarrow\; c^3 = 216 \;\Leftrightarrow\; c = \sqrt[3]{216} = 6.$

Note that $c = -6$ is another solution, since the graphs are the same.