and so Equation 8 becomes

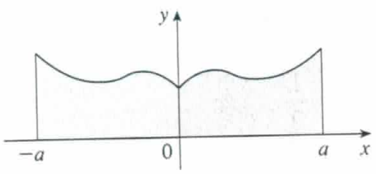
$$\boxed{9} \qquad \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

(a) If f is even, then $f(-u) = f(u)$ so Equation 9 gives

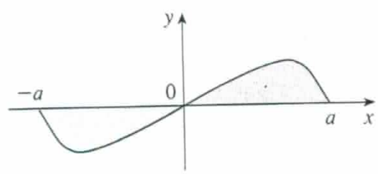
$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$$ □



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

FIGURE 4

Theorem 7 is illustrated by Figure 4. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. Thus part (b) says the integral is 0 because the areas cancel.

EXAMPLE 10 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_0^2 (x^6 + 1)\,dx$$
$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7}$$ □

EXAMPLE 11 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$ □

5.5 EXERCISES

1–6 Evaluate the integral by making the given substitution.

1. $\int e^{-x}\,dx, \quad u = -x$
2. $\int x^3(2 + x^4)^5\,dx, \quad u = 2 + x^4$
3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$
4. $\int \frac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$
5. $\int \cos^3\theta \sin\theta\,d\theta, \quad u = \cos\theta$
6. $\int \frac{\sec^2(1/x)}{x^2}\,dx, \quad u = 1/x$

7–46 Evaluate the indefinite integral.

7. $\int x\sin(x^2)\,dx$
8. $\int x^2(x^3 + 5)^9\,dx$
9. $\int (3x - 2)^{20}\,dx$
10. $\int (3t + 2)^{2.4}\,dt$
11. $\int (x + 1)\sqrt{2x + x^2}\,dx$
12. $\int \frac{x}{(x^2 + 1)^2}\,dx$
13. $\int \frac{dx}{5 - 3x}$
14. $\int e^x \sin(e^x)\,dx$
15. $\int \sin \pi t\,dt$
16. $\int \frac{x}{x^2 + 1}\,dx$
17. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$
18. $\int \sec 2\theta \tan 2\theta\,d\theta$

19. $\int \frac{(\ln x)^2}{x}\,dx$

20. $\int \frac{dx}{ax+b} \quad (a \neq 0)$

21. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

22. $\int \sqrt{x}\sin(1+x^{3/2})\,dx$

23. $\int \cos\theta\,\sin^6\theta\,d\theta$

24. $\int (1+\tan\theta)^5\sec^2\theta\,d\theta$

25. $\int e^x\sqrt{1+e^x}\,dx$

26. $\int e^{\cos t}\sin t\,dt$

27. $\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

28. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$

29. $\int e^{\tan x}\sec^2 x\,dx$

30. $\int \frac{\sin(\ln x)}{x}\,dx$

31. $\int \frac{\cos x}{\sin^2 x}\,dx$

32. $\int \frac{e^x}{e^x+1}\,dx$

33. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

34. $\int \frac{\cos(\pi/x)}{x^2}\,dx$

35. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

36. $\int \frac{\sin x}{1+\cos^2 x}\,dx$

37. $\int \cot x\,dx$

38. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

39. $\int \sec^3 x\tan x\,dx$

40. $\int \sin t\,\sec^2(\cos t)\,dt$

41. $\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1}x}$

42. $\int \frac{x}{1+x^4}\,dx$

43. $\int \frac{1+x}{1+x^2}\,dx$

44. $\int \frac{x^2}{\sqrt{1-x}}\,dx$

45. $\int \frac{x}{\sqrt[4]{x+2}}\,dx$

46. $\int x^3\sqrt{x^2+1}\,dx$

47–50 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

47. $\int x(x^2-1)^3\,dx$

48. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

49. $\int \sin^3 x\cos x\,dx$

50. $\int \tan^2\theta\,\sec^2\theta\,d\theta$

51–70 Evaluate the definite integral.

51. $\int_0^2 (x-1)^{25}\,dx$

52. $\int_0^7 \sqrt{4+3x}\,dx$

53. $\int_0^1 x^2(1+2x^3)^5\,dx$

54. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

55. $\int_0^{\pi} \sec^2(t/4)\,dt$

56. $\int_{1/6}^{1/2} \csc\pi t\,\cot\pi t\,dt$

57. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta$

58. $\int_0^1 xe^{-x^2}\,dx$

59. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

60. $\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

61. $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

62. $\int_0^{\pi/2} \cos x\sin(\sin x)\,dx$

63. $\int_0^a x\sqrt{x^2+a^2}\,dx \quad (a > 0)$

64. $\int_0^a x\sqrt{a^2-x^2}\,dx$

65. $\int_1^2 x\sqrt{x-1}\,dx$

66. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

67. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

68. $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

69. $\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

70. $\int_0^{T/2} \sin(2\pi t/T-\alpha)\,dt$

71–72 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

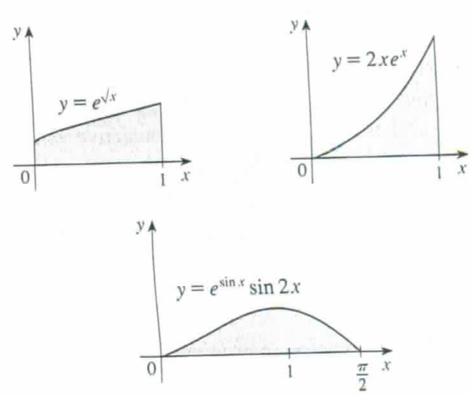
71. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

72. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

73. Evaluate $\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

74. Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

75. Which of the following areas are equal? Why?



76. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\,dt$, over a 24-hour time period?

77. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

78. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

79. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

80. Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t+10)^2}\right) \text{ calculators/week}$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

81. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

82. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

83. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

84. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

85. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

86. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

87. Use Exercise 86 to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

88. (a) If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

(b) Use part (a) to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

5 REVIEW

CONCEPT CHECK

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State both parts of the Fundamental Theorem of Calculus.

4. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

5. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

6. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

7. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

8. State the Substitution Rule. In practice, how do you use it?