

67. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\begin{aligned}\text{coefficient of inequality} &= \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)] dx}{\int_0^1 x dx} \\ &= \frac{\int_0^1 [x - L(x)] dx}{[x^2/2]_0^1} = \frac{\int_0^1 [x - L(x)] dx}{1/2} = 2 \int_0^1 [x - L(x)] dx\end{aligned}$$

- (b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \Rightarrow L(50\%) = L(\frac{1}{2}) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.3958\bar{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned}\text{coefficient of inequality} &= 2 \int_0^1 [x - L(x)] dx = 2 \int_0^1 \left(x - \frac{5}{12}x^2 - \frac{7}{12}x\right) dx = 2 \int_0^1 \left(\frac{5}{12}x - \frac{5}{12}x^2\right) dx \\ &= 2 \int_0^1 \frac{5}{12}(x - x^2) dx = \frac{5}{6} \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{5}{6} \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{5}{6} \left(\frac{1}{6}\right) = \frac{5}{36}\end{aligned}$$

68. (a) From Exercise 4.1.72(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

$$(b) h(125) - h(0) = \int_0^{125} v(t) dt = [0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t]_0^{125} \approx 206,407 \text{ ft}$$

5.5 The Substitution Rule

1. Let $u = -x$. Then $du = -dx$, so $dx = -du$. Thus, $\int e^{-x} dx = \int e^u (-du) = -e^u + C = -e^{-x} + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$$\frac{d}{dx}(-e^{-x} + C) = -[e^{-x}(-1)] = e^{-x}, \text{ the desired result.}$$

2. Let $u = 2 + x^4$. Then $du = 4x^3 dx$ and $x^3 dx = \frac{1}{4} du$,

$$\text{so } \int x^3(2 + x^4)^5 dx = \int u^5 \left(\frac{1}{4} du\right) = \frac{1}{4} \frac{u^6}{6} + C = \frac{1}{24}(2 + x^4)^6 + C.$$

3. Let $u = x^3 + 1$. Then $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$, so

$$\int x^2 \sqrt{x^3 + 1} dx = \int \sqrt{u} \left(\frac{1}{3} du\right) = \frac{1}{3} \frac{u^{3/2}}{3/2} + C = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C = \frac{2}{9}(x^3 + 1)^{3/2} + C.$$

4. Let $u = 1 - 6t$. Then $du = -6 dt$ and $dt = -\frac{1}{6} du$, so

$$\int \frac{dt}{(1 - 6t)^4} = \int \frac{-\frac{1}{6} du}{u^4} = -\frac{1}{6} \int u^{-4} du = -\frac{1}{6} \frac{u^{-3}}{-3} + C = \frac{1}{18u^3} + C = \frac{1}{18(1 - 6t)^3} + C.$$

5. Let $u = \cos \theta$. Then $du = -\sin \theta d\theta$ and $\sin \theta d\theta = -du$, so

$$\int \cos^3 \theta \sin \theta d\theta = \int u^3 (-du) = -\frac{u^4}{4} + C = -\frac{1}{4} \cos^4 \theta + C.$$

6. Let $u = 1/x$. Then $du = -1/x^2 dx$ and $1/x^2 dx = -du$, so

$$\int \frac{\sec^2(1/x)}{x^2} dx = \int \sec^2 u (-du) = -\tan u + C = -\tan(1/x) + C.$$

7. Let $u = x^2$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so $\int x \sin(x^2) dx = \int \sin u (\frac{1}{2} du) = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(x^2) + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2 dx$ and $x^2 dx = \frac{1}{3} du$, so

$$\int x^2(x^3 + 5)^9 dx = \int u^9 \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{1}{10} u^{10} + C = \frac{1}{30}(x^3 + 5)^{10} + C.$$

9. Let $u = 3x - 2$. Then $du = 3 dx$ and $dx = \frac{1}{3} du$, so $\int (3x - 2)^{20} dx = \int u^{20} \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{1}{21} u^{21} + C = \frac{1}{63}(3x - 2)^{21} + C$.

10. Let $u = 3t + 2$. Then $du = 3 dt$ and $dt = \frac{1}{3} du$, so

$$\int (3t + 2)^{2.4} dt = \int u^{2.4} \left(\frac{1}{3} du\right) = \frac{1}{3} \frac{u^{3.4}}{3.4} + C = \frac{1}{10.2}(3t + 2)^{3.4} + C.$$

11. Let $u = 2x + x^2$. Then $du = (2 + 2x) dx = 2(1 + x) dx$ and $(x + 1) dx = \frac{1}{2} du$, so

$$\int (x + 1) \sqrt{2x + x^2} dx = \int \sqrt{u} \left(\frac{1}{2} du\right) = \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{3}(2x + x^2)^{3/2} + C.$$

Or: Let $u = \sqrt{2x + x^2}$. Then $u^2 = 2x + x^2 \Rightarrow 2u du = (2 + 2x) dx \Rightarrow u du = (1 + x) dx$, so

$$\int (x + 1) \sqrt{2x + x^2} dx = \int u \cdot u du = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(2x + x^2)^{3/2} + C.$$

12. Let $u = x^2 + 1$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so

$$\int \frac{x}{(x^2 + 1)^2} dx = \int u^{-2} \left(\frac{1}{2} du\right) = \frac{1}{2} \cdot \frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2 + 1)} + C.$$

13. Let $u = 5 - 3x$. Then $du = -3 dx$ and $dx = -\frac{1}{3} du$, so

$$\int \frac{dx}{5 - 3x} = \int \frac{1}{u} \left(-\frac{1}{3} du\right) = -\frac{1}{3} \ln|u| + C = -\frac{1}{3} \ln|5 - 3x| + C.$$

14. Let $u = e^x$. Then $du = e^x dx$, so $\int e^x \sin(e^x) dx = \int \sin u du = -\cos u + C = -\cos(e^x) + C$.

15. Let $u = \pi t$. Then $du = \pi dt$ and $dt = \frac{1}{\pi} du$, so $\int \sin \pi t dt = \int \sin u \left(\frac{1}{\pi} du\right) = \frac{1}{\pi}(-\cos u) + C = -\frac{1}{\pi} \cos \pi t + C$.

16. Let $u = x^2 + 1$. Then $du = 2x dx$ and $x dx = \frac{1}{2} du$, so

$$\int \frac{x}{x^2 + 1} dx = \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 + 1| + C = \frac{1}{2} \ln(x^2 + 1) + C \quad [\text{since } x^2 + 1 > 0]$$

$$\text{or } \ln(x^2 + 1)^{1/2} + C = \ln \sqrt{x^2 + 1} + C.$$

17. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2) dx = 3(a + bx^2) dx$, so

$$\int \frac{a + bx^2}{\sqrt{3ax + bx^3}} dx = \int \frac{\frac{1}{3} du}{u^{1/2}} = \frac{1}{3} \int u^{-1/2} du = \frac{1}{3} \cdot 2u^2 + C = \frac{2}{3}\sqrt{3ax + bx^3} + C.$$

18. Let $u = 2\theta$. Then $du = 2 d\theta$ and $d\theta = \frac{1}{2} du$, so $\int \sec 2\theta \tan 2\theta d\theta = \int \sec u \tan u \left(\frac{1}{2} du\right) = \frac{1}{2} \sec u + C = \frac{1}{2} \sec 2\theta + C$.

19. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

20. Let $u = ax + b$. Then $du = a dx$ and $dx = (1/a) du$, so

$$\int \frac{dx}{ax + b} = \int \frac{(1/a) du}{u} = \frac{1}{a} \int \frac{1}{u} du = \frac{1}{a} \ln|u| + C = \frac{1}{a} \ln|ax + b| + C.$$

21. Let $u = \sqrt{t}$. Then $du = \frac{dt}{2\sqrt{t}}$ and $\frac{1}{\sqrt{t}} dt = 2 du$, so $\int \frac{\cos \sqrt{t}}{\sqrt{t}} dt = \int \cos u (2 du) = 2 \sin u + C = 2 \sin \sqrt{t} + C$.

22. Let $u = 1 + x^{3/2}$. Then $du = \frac{3}{2}x^{1/2} dx$ and $\sqrt{x} dx = \frac{2}{3} du$, so

$$\int \sqrt{x} \sin(1 + x^{3/2}) dx = \int \sin u \left(\frac{2}{3} du\right) = \frac{2}{3} \cdot (-\cos u) + C = -\frac{2}{3} \cos(1 + x^{3/2}) + C.$$

23. Let $u = \sin \theta$. Then $du = \cos \theta d\theta$, so $\int \cos \theta \sin^6 \theta d\theta = \int u^6 du = \frac{1}{7}u^7 + C = \frac{1}{7} \sin^7 \theta + C$.

24. Let $u = 1 + \tan \theta$. Then $du = \sec^2 \theta d\theta$, so $\int (1 + \tan \theta)^5 \sec^2 \theta d\theta = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}(1 + \tan \theta)^6 + C$.

25. Let $u = 1 + e^x$. Then $du = e^x dx$, so $\int e^x \sqrt{1 + e^x} dx = \int \sqrt{u} du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1 + e^x}$. Then $u^2 = 1 + e^x$ and $2u du = e^x dx$, so

$$\int e^x \sqrt{1 + e^x} dx = \int u \cdot 2u du = \frac{2}{3}u^3 + C = \frac{2}{3}(1 + e^x)^{3/2} + C.$$

26. Let $u = \cos t$. Then $du = -\sin t dt$ and $\sin t dt = -du$, so $\int e^{\cos t} \sin t dt = \int e^u (-du) = -e^u + C = -e^{\cos t} + C$.

27. Let $u = 1 + z^3$. Then $du = 3z^2 dz$ and $z^2 dz = \frac{1}{3} du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}} dz = \int u^{-1/3} \left(\frac{1}{3} du\right) = \frac{1}{3} \cdot \frac{3}{2}u^{2/3} + C = \frac{1}{2}(1+z^3)^{2/3} + C.$$

28. Let $u = \tan^{-1} x$. Then $du = \frac{dx}{1+x^2}$, so $\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C$.

29. Let $u = \tan x$. Then $du = \sec^2 x dx$, so $\int e^{\tan x} \sec^2 x dx = \int e^u du = e^u + C = e^{\tan x} + C$.

30. Let $u = \ln x$. Then $du = (1/x) dx$, so $\int \frac{\sin(\ln x)}{x} dx = \int \sin u du = -\cos u + C = -\cos(\ln x) + C$.

31. Let $u = \sin x$. Then $du = \cos x dx$, so $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{u^2} du = \int u^{-2} du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$

[or $-\csc x + C$].

32. Let $u = e^x + 1$. Then $du = e^x dx$, so $\int \frac{e^x}{e^x + 1} dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x + 1) + C$.

33. Let $u = \cot x$. Then $du = -\csc^2 x dx$ and $\csc^2 x dx = -du$, so

$$\int \sqrt{\cot x} \csc^2 x dx = \int \sqrt{u} (-du) = -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C.$$

34. Let $u = \frac{\pi}{x}$. Then $du = -\frac{\pi}{x^2} dx$ and $\frac{1}{x^2} dx = -\frac{1}{\pi} du$, so

$$\int \frac{\cos(\pi/x)}{x^2} dx = \int \cos u \left(-\frac{1}{\pi} du\right) = -\frac{1}{\pi} \sin u + C = -\frac{1}{\pi} \sin \frac{\pi}{x} + C.$$

35. $\int \frac{\sin 2x}{1 + \cos^2 x} dx = 2 \int \frac{\sin x \cos x}{1 + \cos^2 x} dx = 2I$. Let $u = \cos x$. Then $du = -\sin x dx$, so

$$2I = -2 \int \frac{u du}{1 + u^2} = -2 \cdot \frac{1}{2} \ln(1 + u^2) + C = -\ln(1 + u^2) + C = -\ln(1 + \cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

36. Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin x dx = -du$, so

$$\int \frac{\sin x}{1 + \cos^2 x} dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

37. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$. Let $u = \sin x$. Then $du = \cos x \, dx$, so $\int \cot x \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sin x| + C$.

38. Let $u = 1 + \tan t$. Then $du = \sec^2 t \, dt$, so

$$\int \frac{dt}{\cos^2 t \sqrt{1 + \tan t}} = \int \frac{\sec^2 t \, dt}{\sqrt{1 + \tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} \, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1 + \tan t} + C.$$

39. Let $u = \sec x$. Then $du = \sec x \tan x \, dx$, so

$$\int \sec^3 x \tan x \, dx = \int \sec^2 x (\sec x \tan x) \, dx = \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C.$$

40. Let $u = \cos t$. Then $du = -\sin t \, dt$ and $\sin t \, dt = -du$, so

$$\int \sin t \sec^2(\cos t) \, dt = \int \sec^2 u \cdot (-du) = -\tan u + C = -\tan(\cos t) + C.$$

41. Let $u = \sin^{-1} x$. Then $du = \frac{1}{\sqrt{1-x^2}} \, dx$, so $\int \frac{dx}{\sqrt{1-x^2} \sin^{-1} x} = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sin^{-1} x| + C$.

42. Let $u = x^2$. Then $du = 2x \, dx$, so $\int \frac{x}{1+x^4} \, dx = \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(x^2) + C$.

43. Let $u = 1+x^2$. Then $du = 2x \, dx$, so

$$\begin{aligned} \int \frac{1+x}{1+x^2} \, dx &= \int \frac{1}{1+x^2} \, dx + \int \frac{x}{1+x^2} \, dx = \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} = \tan^{-1} x + \frac{1}{2}\ln|u| + C \\ &= \tan^{-1} x + \frac{1}{2}\ln|1+x^2| + C = \tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C \quad [\text{since } 1+x^2 > 0]. \end{aligned}$$

44. Let $u = 1-x$. Then $x = 1-u$ and $dx = -du$, so

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-x}} \, dx &= \int \frac{(1-u)^2}{\sqrt{u}} (-du) = - \int \frac{1-2u+u^2}{\sqrt{u}} \, du = - \int (u^{-1/2} - 2u^{1/2} + u^{3/2}) \, du \\ &= - \left(2u^{1/2} - 2 \cdot \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} \right) + C = -2\sqrt{1-x} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C \end{aligned}$$

45. Let $u = x+2$. Then $du = dx$, so

$$\begin{aligned} \int \frac{x}{\sqrt[4]{x+2}} \, dx &= \int \frac{u-2}{\sqrt[4]{u}} \, du = \int (u^{3/4} - 2u^{-1/4}) \, du = \frac{4}{7}u^{7/4} - 2 \cdot \frac{4}{3}u^{3/4} + C \\ &= \frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C \end{aligned}$$

46. Let $u = x^2 + 1$ [so $x^2 = u-1$]. Then $du = 2x \, dx$ and $x \, dx = \frac{1}{2} \, du$, so

$$\begin{aligned} \int x^3 \sqrt{x^2+1} \, dx &= \int x^2 \sqrt{x^2+1} x \, dx = \int (u-1)\sqrt{u} \left(\frac{1}{2} \, du \right) = \frac{1}{2} \int (u^{3/2} - u^{1/2}) \, du \\ &= \frac{1}{2} \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right) + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C. \end{aligned}$$

Or: Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1 \Rightarrow 2u \, du = 2x \, dx \Rightarrow u \, du = x \, dx$, so

$$\begin{aligned} \int x^3 \sqrt{x^2+1} \, dx &= \int x^2 \sqrt{x^2+1} x \, dx = \int (u^2-1)u \cdot u \, du = \int (u^4 - u^2) \, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C. \end{aligned}$$

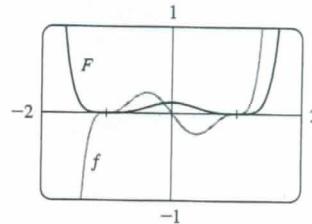
Note: This answer can be written as $\frac{1}{15}\sqrt{x^2+1}(3x^4 + x^2 - 2) + C$.

In Exercises 47–50, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

47. $f(x) = x(x^2 - 1)^3$. $u = x^2 - 1 \Rightarrow du = 2x dx$, so

$$\int x(x^2 - 1)^3 dx = \int u^3 \left(\frac{1}{2} du\right) = \frac{1}{8}u^4 + C = \frac{1}{8}(x^2 - 1)^4 + C$$

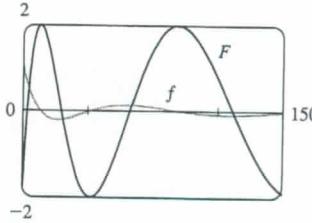
Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).



48. $f(x) = \frac{\sin \sqrt{x}}{\sqrt{x}}$. $u = \sqrt{x} \Rightarrow du = \frac{1}{2}x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx$, so

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin u (2 du) = -2 \cos \sqrt{x} + C$$

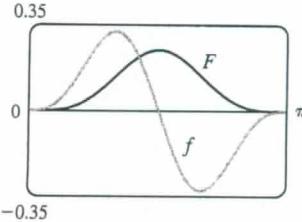
Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).



49. $f(x) = \sin^3 x \cos x$. $u = \sin x \Rightarrow du = \cos x dx$, so

$$\int \sin^3 x \cos x dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$$

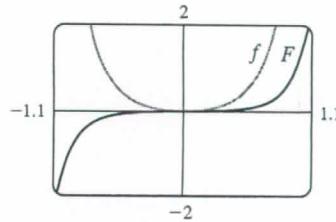
Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π , so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



50. $f(\theta) = \tan^2 \theta \sec^2 \theta$. $u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$, so

$$\int \tan^2 \theta \sec^2 \theta d\theta = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 \theta + C$$

Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



51. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25} dx = \int_{-1}^1 u^{25} du = 0$ by Theorem 7(b), since $f(u) = u^{25}$ is an odd function.

52. Let $u = 4 + 3x$, so $du = 3 dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x} dx = \int_4^{25} \sqrt{u} \left(\frac{1}{3} du\right) = \frac{1}{3} \left[\frac{u^{3/2}}{3/2} \right]_4^{25} = \frac{2}{9}(25^{3/2} - 4^{3/2}) = \frac{2}{9}(125 - 8) = \frac{234}{9} = 26.$$

53. Let $u = 1 + 2x^3$, so $du = 6x^2 dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2 (1 + 2x^3)^5 dx = \int_1^3 u^5 \left(\frac{1}{6} du\right) = \frac{1}{6} \left[\frac{1}{6} u^6 \right]_1^3 = \frac{1}{36}(3^6 - 1^6) = \frac{1}{36}(729 - 1) = \frac{728}{36} = \frac{182}{9}.$$

54. Let $u = x^2$, so $du = 2x dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\int_0^{\sqrt{\pi}} x \cos(x^2) dx = \int_0^{\pi} \cos u \left(\frac{1}{2} du\right) = \frac{1}{2} [\sin u]_0^{\pi} = \frac{1}{2}(\sin \pi - \sin 0) = \frac{1}{2}(0 - 0) = 0.$$

55. Let $u = t/4$, so $du = \frac{1}{4} dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$$\int_0^\pi \sec^2(t/4) dt = \int_0^{\pi/4} \sec^2 u (4 du) = 4[\tan u]_0^{\pi/4} = 4(\tan \frac{\pi}{4} - \tan 0) = 4(1 - 0) = 4.$$

56. Let $u = \pi t$, so $du = \pi dt$. When $t = \frac{1}{6}$, $u = \frac{\pi}{6}$; when $t = \frac{1}{2}$, $u = \frac{\pi}{2}$. Thus,

$$\int_{1/6}^{1/2} \csc \pi t \cot \pi t dt = \int_{\pi/6}^{\pi/2} \csc u \cot u (\frac{1}{\pi} du) = \frac{1}{\pi} [-\csc u]_{\pi/6}^{\pi/2} = -\frac{1}{\pi}(1 - 2) = \frac{1}{\pi}.$$

57. $\int_{-\pi/6}^{\pi/6} \tan^3 \theta d\theta = 0$ by Theorem 7(b), since $f(\theta) = \tan^3 \theta$ is an odd function.

58. Let $u = -x^2$, so $du = -2x dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2} dx = \int_0^{-1} e^u (-\frac{1}{2} du) = -\frac{1}{2}[e^u]_0^{-1} = -\frac{1}{2}(e^{-1} - e^0) = \frac{1}{2}(1 - 1/e).$$

59. Let $u = 1/x$, so $du = -1/x^2 dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^{1/2} e^u (-du) = -[e^u]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

60. $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} dx = 0$ by Theorem 7(b), since $f(x) = \frac{x^2 \sin x}{1 + x^6}$ is an odd function.

61. Let $u = 1 + 2x$, so $du = 2 dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3} (\frac{1}{2} du) = \left[\frac{1}{2} \cdot 3u^{1/3} \right]_1^{27} = \frac{3}{2}(3 - 1) = 3.$$

62. Let $u = \sin x$, so $du = \cos x dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x \sin(\sin x) dx = \int_0^1 \sin u du = [-\cos u]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

63. Let $u = x^2 + a^2$, so $du = 2x dx$ and $x dx = \frac{1}{2} du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x \sqrt{x^2 + a^2} dx = \int_{a^2}^{2a^2} u^{1/2} (\frac{1}{2} du) = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{a^2}^{2a^2} = \left[\frac{1}{3} u^{3/2} \right]_{a^2}^{2a^2} = \frac{1}{3} [(2a^2)^{3/2} - (a^2)^{3/2}] = \frac{1}{3} (2\sqrt{2} - 1)a^3$$

64. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x \sqrt{a^2 - x^2} dx = \int_{a^2}^0 u^{1/2} (-\frac{1}{2} du) = \frac{1}{2} \int_0^{a^2} u^{1/2} du = \frac{1}{2} \cdot \left[\frac{2}{3} u^{3/2} \right]_0^{a^2} = \frac{1}{3} a^3.$$

65. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x \sqrt{x-1} dx = \int_0^1 (u+1) \sqrt{u} du = \int_0^1 (u^{3/2} + u^{1/2}) du = \left[\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}.$$

66. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u - 1)$ and $du = 2 dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\begin{aligned} \int_0^4 \frac{x dx}{\sqrt{1+2x}} &= \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \frac{du}{2} = \frac{1}{4} \int_1^9 (u^{1/2} - u^{-1/2}) du = \frac{1}{4} \left[\frac{2}{3} u^{3/2} - 2u^{1/2} \right]_1^9 = \frac{1}{4} \cdot \frac{2}{3} \left[u^{3/2} - 3u^{1/2} \right]_1^9 \\ &= \frac{1}{6} [(27 - 9) - (1 - 3)] = \frac{20}{6} = \frac{10}{3} \end{aligned}$$

67. Let $u = \ln x$, so $du = \frac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$, $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2} du = 2[u^{1/2}]_1^4 = 2(2 - 1) = 2.$$

68. Let $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \left[\frac{u^2}{2} \right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

69. Let $u = e^z + z$, so $du = (e^z + 1) dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z + 1}{e^z + z} dz = \int_1^{e+1} \frac{1}{u} du = [\ln|u|]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

70. Let $u = \frac{2\pi t}{T} - \alpha$, so $du = \frac{2\pi}{T} dt$. When $t = 0$, $u = -\alpha$; when $t = \frac{T}{2}$, $u = \pi - \alpha$. Thus,

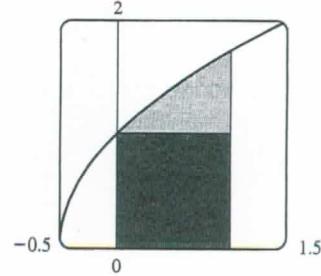
$$\begin{aligned} \int_0^{\pi/2} \sin\left(\frac{2\pi t}{T} - \alpha\right) dt &= \int_{-\alpha}^{\pi-\alpha} \sin u \left(\frac{T}{2\pi} du\right) = \frac{T}{2\pi} [-\cos u]_{-\alpha}^{\pi-\alpha} = -\frac{T}{2\pi} [\cos(\pi - \alpha) - \cos(-\alpha)] \\ &= -\frac{T}{2\pi} (-\cos \alpha - \cos \alpha) = -\frac{T}{2\pi} (-2 \cos \alpha) = \frac{T}{\pi} \cos \alpha \end{aligned}$$

71. From the graph, it appears that the area under the curve is about

$1 + (\text{a little more than } \frac{1}{2} \cdot 1 \cdot 0.7)$, or about 1.4. The exact area is given by

$A = \int_0^1 \sqrt{2x+1} dx$. Let $u = 2x+1$, so $du = 2 dx$. The limits change to $2 \cdot 0 + 1 = 1$ and $2 \cdot 1 + 1 = 3$, and

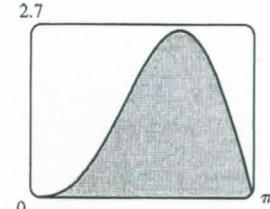
$$A = \int_1^3 \sqrt{u} \left(\frac{1}{2} du\right) = \frac{1}{2} \left[\frac{2}{3} u^{3/2}\right]_1^3 = \frac{1}{3} (3\sqrt{3} - 1) = \sqrt{3} - \frac{1}{3} \approx 1.399.$$



72. From the graph, it appears that the area under the curve is almost $\frac{1}{2} \cdot \pi \cdot 2.6$,

or about 4. The exact area is given by

$$\begin{aligned} A &= \int_0^\pi (2 \sin x - \sin 2x) dx = -2 [\cos x]_0^\pi - \int_0^\pi \sin 2x dx \\ &= -2(-1 - 1) - 0 = 4 \end{aligned}$$



Note: $\int_0^\pi \sin 2x dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^\pi \sin 2x dx = -\int_0^{\pi/2} \sin 2x dx$.

73. First write the integral as a sum of two integrals:

$$I = \int_{-2}^2 (x+3)\sqrt{4-x^2} dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2} dx + \int_{-2}^2 3\sqrt{4-x^2} dx. I_1 = 0$$

since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3 \cdot \frac{1}{2}(\pi \cdot 2^2) = 6\pi$.

74. Let $u = x^2$. Then $du = 2x dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so

$$I = \int_0^1 x\sqrt{1-x^4} dx = \frac{1}{2} \int_0^1 \sqrt{1-u^2} du. \text{ But this integral can be interpreted as the area of a quarter-circle with radius 1.}$$

$$\text{So } I = \frac{1}{2} \cdot \frac{1}{4} (\pi \cdot 1^2) = \frac{1}{8}\pi.$$

75. First Figure Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$$A_1 = \int_0^1 e^{\sqrt{x}} dx = \int_0^1 e^u (2u du) = 2 \int_0^1 ue^u du.$$

$$\text{Second Figure } A_2 = \int_0^1 2xe^x dx = 2 \int_0^1 ue^u du.$$

Third Figure Let $u = \sin x$, so $du = \cos x dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$A_3 = \int_0^{\pi/2} e^{\sin x} \sin 2x dx = \int_0^{\pi/2} e^{\sin x} (2 \sin x \cos x) dx = \int_0^1 e^u (2u du) = 2 \int_0^1 ue^u du.$$

Since $A_1 = A_2 = A_3$, all three areas are equal.

76. Let $u = \frac{\pi t}{12}$. Then $du = \frac{\pi}{12} dt$ and

$$\begin{aligned} \int_0^{24} R(t) dt &= \int_0^{24} \left[85 - 0.18 \cos\left(\frac{\pi t}{12}\right) \right] dt = \int_0^{2\pi} (85 - 0.18 \cos u) \left(\frac{12}{\pi} du \right) = \frac{12}{\pi} [85u - 0.18 \sin u]_0^{2\pi} \\ &= \frac{12}{\pi} [(85 \cdot 2\pi - 0) - (0 - 0)] = 2040 \text{ kcal} \end{aligned}$$

77. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$$\begin{aligned} \int_0^{60} r(t) dt &= \int_0^{60} 100e^{-0.01t} dt \quad [u = -0.01t, du = -0.01dt] \\ &= 100 \int_0^{-0.6} e^u (-100 du) = -10,000 [e^u]_0^{-0.6} = -10,000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ liters} \end{aligned}$$

78. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$,

$\int_0^3 r(t) dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$\begin{aligned} n(3) &= 400 + \int_0^3 r(t) dt = 400 + \int_0^3 ae^{bt} dt = 400 + \frac{a}{b} [e^{bt}]_0^3 = 400 + \frac{a}{b} (e^{3b} - 1) \\ &\approx 400 + 11,313 = 11,713 \text{ bacteria} \end{aligned}$$

79. The volume of inhaled air in the lungs at time t is

$$\begin{aligned} V(t) &= \int_0^t f(u) du = \int_0^t \frac{1}{2} \sin\left(\frac{2\pi}{5} u\right) du = \int_0^{2\pi t/5} \frac{1}{2} \sin v \left(\frac{5}{2\pi} dv \right) \quad [\text{substitute } v = \frac{2\pi}{5}u, dv = \frac{2\pi}{5} du] \\ &= \frac{5}{4\pi} [-\cos v]_0^{2\pi t/5} = \frac{5}{4\pi} [-\cos(\frac{2\pi}{5}t) + 1] = \frac{5}{4\pi} [1 - \cos(\frac{2\pi}{5}t)] \text{ liters} \end{aligned}$$

80. Number of calculators = $x(4) - x(2) = \int_2^4 5000 [1 - 100(t+10)^{-2}] dt$

$$= 5000 [t + 100(t+10)^{-1}]_2^4 = 5000 [(4 + \frac{100}{14}) - (2 + \frac{100}{12})] \approx 4048$$

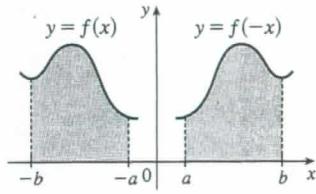
81. Let $u = 2x$. Then $du = 2 dx$, so $\int_0^2 f(2x) dx = \int_0^4 f(u)(\frac{1}{2} du) = \frac{1}{2} \int_0^4 f(u) du = \frac{1}{2}(10) = 5$.

82. Let $u = x^2$. Then $du = 2x dx$, so $\int_0^3 xf(x^2) dx = \int_0^9 f(u)(\frac{1}{2} du) = \frac{1}{2} \int_0^9 f(u) du = \frac{1}{2}(4) = 2$.

83. Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x) dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u) du = \int_{-b}^{-a} f(x) dx$$

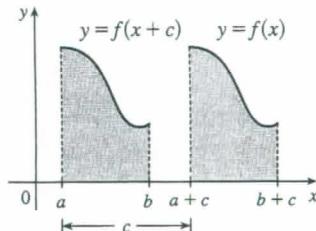
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f , and the limits of integration, about the y -axis.



84. Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c) dx = \int_{a+c}^{b+c} f(u) du = \int_{a+c}^{b+c} f(x) dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f , and the limits of integration, by a distance c .



85. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a (1-x)^b dx = \int_1^0 (1-u)^a u^b (-du) = \int_0^1 u^b (1-u)^a du = \int_0^1 x^b (1-x)^a dx.$$

86. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\begin{aligned} \int_0^\pi x f(\sin x) dx &= - \int_\pi^0 (\pi - u) f(\sin(\pi - u)) du = \int_0^\pi (\pi - u) f(\sin u) du \\ &= \pi \int_0^\pi f(\sin u) du - \int_0^\pi u f(\sin u) du = \pi \int_0^\pi f(\sin x) dx - \int_0^\pi x f(\sin x) dx \Rightarrow \end{aligned}$$

$$2 \int_0^\pi x f(\sin x) dx = \pi \int_0^\pi f(\sin x) dx \Rightarrow \int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx.$$

87. $\frac{x \sin x}{1 + \cos^2 x} = x \cdot \frac{\sin x}{2 - \sin^2 x} = x f(\sin x)$, where $f(t) = \frac{t}{2 - t^2}$. By Exercise 86,

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

Let $u = \cos x$. Then $du = -\sin x dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\begin{aligned} \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx &= -\frac{\pi}{2} \int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2} \int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2} [\tan^{-1} u]_{-1}^1 \\ &= \frac{\pi}{2} [\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{\pi^2}{4} \end{aligned}$$

88. (a) $\int_0^{\pi/2} f(\cos x) dx = \int_0^{\pi/2} f[\sin(\frac{\pi}{2} - x)] dx \quad [u = \frac{\pi}{2} - x, du = -dx]$
 $= \int_{\pi/2}^0 f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u) du = \int_0^{\pi/2} f(\sin x) dx$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

- (b) In part (a), take $f(x) = x^2$, so $\int_0^{\pi/2} \cos^2 x dx = \int_0^{\pi/2} \sin^2 x dx$. Now

$$\int_0^{\pi/2} \cos^2 x dx + \int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x) dx = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2},$$

$$\text{so } 2 \int_0^{\pi/2} \cos^2 x dx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \cos^2 x dx = \frac{\pi}{4} \quad \left[= \int_0^{\pi/2} \sin^2 x dx \right].$$