38. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$

$$= \left(c_0 + c_1x + c_2x^2 + c_3x^3\right)\left(1 + x^4 + x^8 + \cdots + x^{4n-4}\right) \to \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4} \text{ as } n \to \infty$$

[by (11.2.4) with $r = x^4$] for $|x^4| < 1 \quad \Leftrightarrow \quad |x| < 1$. Also s_{4n}, s_{4n+1}, s_{4n+2} have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

39. We use the Root Test on the series $\sum c_nx^n$. We need $\lim\limits_{n\to\infty} \sqrt[n]{|c_nx^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

40. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n(x - a)^n$, we find that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty} \frac{|x-a|}{|c_n/c_{n+1}|}\ (*) = \frac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n+1}|}$$ (if $\lim\limits_{n\to\infty} |c_n/c_{n+1}| \neq 0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n+1}|} < 1 \quad \Leftrightarrow \quad |x - a| < \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$. Thus, $R = \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$. If $\lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right| = 0$ and $|x - a| \neq 0$, then $(*)$ shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$.

41. For $2 < x < 3$, $\sum c_nx^n$ diverges and $\sum d_nx^n$ converges. By Exercise 11.2.69, $\sum(c_n + d_n)\,x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum(c_n + d_n)\,x^n$ is 2.

42. Since $\sum c_nx^n$ converges whenever $|x| < R$, $\sum c_nx^{2n} = \sum c_n\left(x^2\right)^n$ converges whenever $|x^2| < R \quad \Leftrightarrow \quad |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

11.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_nx^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} nc_nx^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_nx^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1}x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^nx^n$ with $|-x| < 1 \quad \Leftrightarrow \quad |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{3}{1-x^4} = 3\left(\dfrac{1}{1-x^4}\right) = 3(1 + x^4 + x^8 + x^{12} + \cdots) = 3\sum\limits_{n=0}^{\infty}(x^4)^n = \sum\limits_{n=0}^{\infty} 3x^{4n}$

with $|x^4| < 1 \quad \Leftrightarrow \quad |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

$\left[\text{Note that } 3\sum\limits_{n=0}^{\infty}(x^4)^n \text{ converges} \quad \Leftrightarrow \quad \sum\limits_{n=0}^{\infty}(x^4)^n \text{ converges, so the appropriate condition [from equation (1)] is } |x^4| < 1.\right]$

5. $f(x) = \dfrac{2}{3-x} = \dfrac{2}{3}\left(\dfrac{1}{1-x/3}\right) = \dfrac{2}{3}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n$ or, equivalently, $2\sum\limits_{n=0}^{\infty}\dfrac{1}{3^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{3}\right| < 1$, that is, when $|x| < 3$, so $R = 3$ and $I = (-3, 3)$.

6. $f(x) = \dfrac{1}{x+10} = \dfrac{1}{10}\left(\dfrac{1}{1-(-x/10)}\right) = \dfrac{1}{10}\sum\limits_{n=0}^{\infty}\left(-\dfrac{x}{10}\right)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{10^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{10}\right| < 1$, that is, when $|x| < 10$, so $R = 10$ and $I = (-10, 10)$.

7. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \Leftrightarrow \dfrac{|x^2|}{9} < 1 \Leftrightarrow |x|^2 < 9 \Leftrightarrow |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

8. $f(x) = \dfrac{x}{2x^2+1} = x\left(\dfrac{1}{1-(-2x^2)}\right) = x\sum\limits_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when $|-2x^2| < 1 \Rightarrow |x^2| < \frac{1}{2} \Rightarrow |x| < \dfrac{1}{\sqrt{2}}$, so $R = \dfrac{1}{\sqrt{2}}$ and $I = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

9. $f(x) = \dfrac{1+x}{1-x} = (1+x)\left(\dfrac{1}{1-x}\right) = (1+x)\sum\limits_{n=0}^{\infty}x^n = \sum\limits_{n=0}^{\infty}x^n + \sum\limits_{n=0}^{\infty}x^{n+1} = 1 + \sum\limits_{n=1}^{\infty}x^n + \sum\limits_{n=1}^{\infty}x^n = 1 + 2\sum\limits_{n=1}^{\infty}x^n$.

The series converges when $|x| < 1$, so $R = 1$ and $I = (-1, 1)$.

A second approach: $f(x) = \dfrac{1+x}{1-x} = \dfrac{-(1-x)+2}{1-x} = -1 + 2\left(\dfrac{1}{1-x}\right) = -1 + 2\sum\limits_{n=0}^{\infty}x^n = 1 + 2\sum\limits_{n=1}^{\infty}x^n$.

A third approach:

$$f(x) = \frac{1+x}{1-x} = (1+x)\left(\frac{1}{1-x}\right) = (1+x)(1+x+x^2+x^3+\cdots)$$
$$= (1+x+x^2+x^3+\cdots) + (x+x^2+x^3+x^4+\cdots) = 1 + 2x + 2x^2 + 2x^3 + \cdots = 1 + 2\sum_{n=1}^{\infty}x^n.$$

10. $f(x) = \dfrac{x^2}{a^3-x^3} = \dfrac{x^2}{a^3}\cdot\dfrac{1}{1-x^3/a^3} = \dfrac{x^2}{a^3}\sum\limits_{n=0}^{\infty}\left(\dfrac{x^3}{a^3}\right)^n = \sum\limits_{n=0}^{\infty}\dfrac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \Leftrightarrow |x^3| < |a^3| \Leftrightarrow |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

11. $f(x) = \dfrac{3}{x^2-x-2} = \dfrac{3}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} \Rightarrow 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and $x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1-(x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$
$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in (-1, 1) = I$.

12. $f(x) = \dfrac{x+2}{2x^2-x-1} = \dfrac{x+2}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1} \;\Rightarrow\; x+2 = A(x-1)+B(2x+1)$. Let $x=1$ to get $3 = 3B \;\Rightarrow\; B=1$ and $x=-\frac{1}{2}$ to get $\frac{3}{2} = -\frac{3}{2}A \;\Rightarrow\; A=-1$. Thus,

$$\frac{x+2}{2x^2-x-1} = \frac{-1}{2x+1} + \frac{1}{x-1} = -1\left(\frac{1}{1-(-2x)}\right) - 1\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}(-2x)^n - \sum_{n=0}^{\infty}x^n$$
$$= -\sum_{n=0}^{\infty}\left[(-2)^n+1\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and the second converges for $(-1,1)$. Thus, the sum converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R=1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\displaystyle\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R=1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2\cdot\dfrac{1}{(1+x)^3} = x^2\cdot\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R=1$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R=1$], so

$f(x) = \ln(1+x) = \displaystyle\int\frac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$

[$C=0$ since $f(0) = \ln 1 = 0$], with $R=1$

(b) $f(x) = x\ln(1+x) = x\left[\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$ with $R=1$.

(c) $f(x) = \ln(x^2+1) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with $R=1$.

15. $f(x) = \ln(5-x) = -\displaystyle\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,5^n}$

Putting $x=0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \;\Leftrightarrow\; |x| < 5$, so $R = 5$.

16. We know that $\dfrac{1}{1-2x} = \sum\limits_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum\limits_{n=1}^{\infty} 2^n n x^{n-1} = \sum\limits_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} 2^n (n+1) x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2}(n-1)x^n,$$

with $R = \frac{1}{2}$.

17. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2} \sum\limits_{n=0}^{\infty} \left(\dfrac{x}{2}\right)^n = \sum\limits_{n=0}^{\infty} \dfrac{1}{2^{n+1}} x^n$ for $\left|\dfrac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. Now

$$\frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n\right) = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n. \text{ So}$$

$$f(x) = \frac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3} \text{ or } \sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n$$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

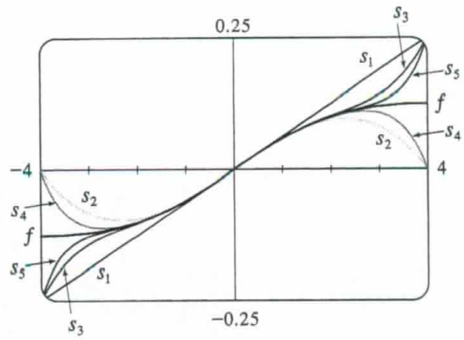
18. From Example 7, $g(x) = \arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3, \text{ so } R = 3.$$

19. $f(x) = \dfrac{x}{x^2+16} = \dfrac{x}{16}\left(\dfrac{1}{1-(-x^2/16)}\right) = \dfrac{x}{16} \sum\limits_{n=0}^{\infty} \left(-\dfrac{x^2}{16}\right)^n = \dfrac{x}{16} \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{1}{16^n} x^{2n} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{1}{16^{n+1}} x^{2n+1}$.

The series converges when $\left|-x^2/16\right| < 1 \Leftrightarrow x^2 < 16 \Leftrightarrow |x| < 4$, so $R = 4$. The partial sums are $s_1 = \dfrac{x}{16}$,

$s_2 = s_1 - \dfrac{x^3}{16^2}$, $s_3 = s_2 + \dfrac{x^5}{16^3}$, $s_4 = s_3 - \dfrac{x^7}{16^4}$, $s_5 = s_4 + \dfrac{x^9}{16^5}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent.



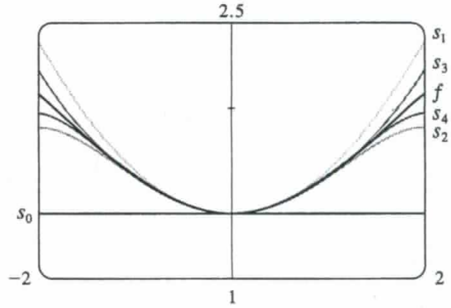
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-4, 4)$.

20. $f(x) = \ln(x^2+4) \Rightarrow f'(x) = \dfrac{2x}{x^2+4} = \dfrac{2x}{4}\left(\dfrac{1}{1-(-x^2/4)}\right) = \dfrac{x}{2} \sum\limits_{n=0}^{\infty} \left(-\dfrac{x^2}{4}\right)^n = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2^{2n+1}}$,

so $f(x) = \displaystyle\int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+1}(2n+2)} = \ln 4 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(n+1)2^{2n+2}}$

$[f(0) = \ln 4$, so $C = \ln 4]$. The series converges when $\left|-x^2/4\right| < 1 \Leftrightarrow x^2 < 4 \Leftrightarrow |x| < 2$, so $R = 2$. If

$x = \pm 2$, then $f(x) = \ln 4 + \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$, which converges by the Alternating Series Test. The partial sums are $s_0 = \ln 4$ [≈ 1.39], $s_1 = s_0 + \frac{x^2}{4}$, $s_2 = s_1 - \frac{x^4}{2 \cdot 2^4}$, $s_3 = s_2 + \frac{x^6}{3 \cdot 2^6}$, $s_4 = s_3 - \frac{x^8}{4 \cdot 2^8}$, ...



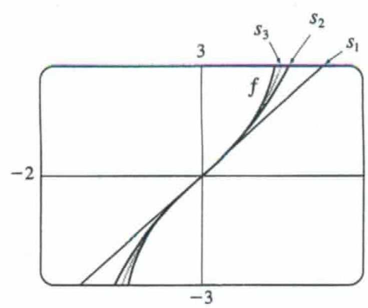
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-2, 2]$.

21. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x} = \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int [(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)]\, dx$$

$$= \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \frac{1}{n}$. The partial sums are $s_1 = \frac{2x}{1}$, $s_2 = s_1 + \frac{2x^3}{3}$, $s_3 = s_2 + \frac{2x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

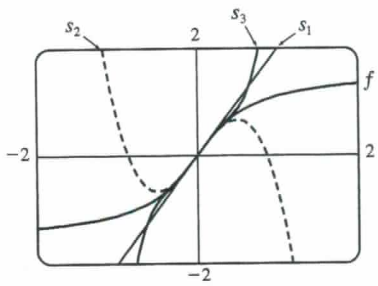
22. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \qquad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0]$$

The series converges when $|4x^2| < 1 \;\Leftrightarrow\; |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test. The partial sums are

$s_1 = \dfrac{2x}{1}$, $s_2 = s_1 - \dfrac{2^3 x^3}{3}$, $s_3 = s_2 + \dfrac{2^5 x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

23. $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t \sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \quad \Rightarrow \quad \int \dfrac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty} \dfrac{t^{8n+2}}{8n+2}$. The series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1 \quad \Leftrightarrow \quad |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \dfrac{t}{1-t^8}\,dt$ also has $R = 1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \dfrac{t^n}{n}$ for $|t| < 1$, so $\dfrac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \dfrac{t^{n-1}}{n}$ and $\int \dfrac{\ln(1-t)}{t}\,dt = C - \sum_{n=1}^{\infty} \dfrac{t^n}{n^2}$.

By Theorem 2, $R = 1$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$x - \tan^{-1} x = x - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots\right) = \dfrac{x^3}{3} - \dfrac{x^5}{5} + \dfrac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n+1}}{2n+1}$ and

$\dfrac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n-2}}{2n+1}$, so

$\int \dfrac{x - \tan^{-1} x}{x^3}\,dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n-1}}{4n^2-1}$. By Theorem 2, $R = 1$.

26. By Example 7, $\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

27. $\dfrac{1}{1+x^5} = \dfrac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n} \quad \Rightarrow$

$\int \dfrac{1}{1+x^5}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{5n+1}}{5n+1}$. Thus,

$I = \int_0^{0.2} \dfrac{1}{1+x^5}\,dx = \left[x - \dfrac{x^6}{6} + \dfrac{x^{11}}{11} - \cdots\right]_0^{0.2} = 0.2 - \dfrac{(0.2)^6}{6} + \dfrac{(0.2)^{11}}{11} - \cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$. So $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$ to six decimal places.

28. From Example 6, we know $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, so

$$\ln(1+x^4) = \ln[1-(-x^4)] = -\sum_{n=1}^{\infty} \frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} \quad \Rightarrow$$

$$\int \ln(1+x^4)\,dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n}\,dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n+1}}{n(4n+1)}. \text{ Thus,}$$

$$I = \int_0^{0.4} \ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots\right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots.$$

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$.

So $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.9)^{13}/39 \approx 0.002034$ to six decimal places.

29. We substitute $3x$ for x in Example 7, and find that

$$\int x \arctan(3x)\,dx = \int x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+2}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+3}}{(2n+1)(2n+3)}$$

So
$$\int_0^{0.1} x \arctan(3x)\,dx = \left[\frac{3x^3}{1\cdot 3} - \frac{3^3 x^5}{3\cdot 5} + \frac{3^5 x^7}{5\cdot 7} - \frac{3^7 x^9}{7\cdot 9} + \cdots\right]_0^{0.1}$$
$$= \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} - \frac{2187}{63\times 10^9} + \cdots.$$

The series is alternating, so if we use three terms, the error is at most $\frac{2187}{63\times 10^9} \approx 3.5\times 10^{-8}$. So

$$\int_0^{0.1} x \arctan(3x)\,dx \approx \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} \approx 0.000\,983 \text{ to six decimal places.}$$

30. $$\int_0^{0.3} \frac{x^2}{1+x^4}\,dx = \int_0^{0.3} x^2 \sum_{n=0}^{\infty} (-1)^n x^{4n}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+3}}{4n+3}\right]_0^{0.3} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 3^{4n+3}}{(4n+3)10^{4n+3}}$$
$$= \frac{3^3}{3\times 10^3} - \frac{3^7}{7\times 10^7} + \frac{3^{11}}{11\times 10^{11}} - \cdots$$

The series is alternating, so if we use only two terms, the error is at most $\frac{3^{11}}{11\times 10^{11}} \approx 0.000\,000\,16$. So, to six decimal places, $\int_0^{0.3} \frac{x^2}{1+x^4}\,dx \approx \frac{3^3}{3\times 10^3} - \frac{3^7}{7\times 10^7} \approx 0.008\,969$.

31. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, with $x = -0.1$, we have

$\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$.

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad \text{[substituting } n+1 \text{ for } n\text{]}$$
$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad \Rightarrow \quad f''(x) + f(x) = 0.$$

33. (a) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$\begin{aligned} x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{-1}2^2n^2x^{2n}}{2^{2n}(n!)^2} \\ &= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2n^2}{2^{2n}(n!)^2}\right] x^{2n} \\ &= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0 \end{aligned}$$

(b) $$\begin{aligned} \int_0^1 J_0(x)\,dx &= \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx \\ &= \left[x - \frac{x^3}{3 \cdot 4} + \frac{x^5}{5 \cdot 64} - \frac{x^7}{7 \cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots \end{aligned}$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places,

$\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

34. (a) $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\,x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and $J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\,x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}$.

$$\begin{aligned} &x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &\qquad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &\qquad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n-1 \\ \text{in the third term}\end{array}\right] \\ &= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0 \end{aligned}$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \Rightarrow$

$$\begin{aligned} J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2(n+1)x^{2n+1}}{2^{2n+2}[(n+1)!]^2} \qquad [\text{Replace } n \text{ with } n+1] \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \qquad [\text{cancel 2 and } n+1\text{; take } -1 \text{ outside sum}] \quad = -J_1(x) \end{aligned}$$

35. (a) $f(x)=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} \quad\Rightarrow\quad f'(x)=\sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n!}=\sum\limits_{n=1}^{\infty}\dfrac{x^{n-1}}{(n-1)!}=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}=f(x)$

(b) By Theorem 9.4.2, the only solution to the differential equation $df(x)/dx=f(x)$ is $f(x)=Ke^x$, but $f(0)=1$, so $K=1$ and $f(x)=e^x$.

Or: We could solve the equation $df(x)/dx=f(x)$ as a separable differential equation.

36. $\dfrac{|\sin nx|}{n^2}\le\dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n^2}$ converges by the Comparison Test. $\dfrac{d}{dx}\left(\dfrac{\sin nx}{n^2}\right)=\dfrac{\cos nx}{n}$, so when $x=2k\pi$ [k an integer], $\sum\limits_{n=1}^{\infty}f_n'(x)=\sum\limits_{n=1}^{\infty}\dfrac{\cos(2kn\pi)}{n}=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which diverges [harmonic series]. $f_n''(x)=-\sin nx$, so $\sum\limits_{n=1}^{\infty}f_n''(x)=-\sum\limits_{n=1}^{\infty}\sin nx$, which converges only if $\sin nx=0$, or $x=k\pi$ [k an integer].

37. If $a_n=\dfrac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{x^n}\right|=|x|\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2=|x|<1$ for convergence, so $R=1$. When $x=\pm1$, $\sum\limits_{n=1}^{\infty}\left|\dfrac{x^n}{n^2}\right|=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p=2>1$), so the interval of convergence for f is $[-1,1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x)=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2}\quad\Rightarrow\quad f'(x)=\sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n^2}=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}$, and this series diverges for $x=1$ (harmonic series) and converges for $x=-1$ (Alternating Series Test), so the interval of convergence is $[-1,1)$. $f''(x)=\sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty}\dfrac{n}{n+1}=1\neq0$, so its interval of convergence is $(-1,1)$.

38. (a) $\sum\limits_{n=1}^{\infty}nx^{n-1}=\sum\limits_{n=0}^{\infty}\dfrac{d}{dx}x^n=\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}x^n\right]=\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right]=-\dfrac{1}{(1-x)^2}(-1)=\dfrac{1}{(1-x)^2}$, $|x|<1$.

(b) (i) $\sum\limits_{n=1}^{\infty}nx^n=x\sum\limits_{n=1}^{\infty}nx^{n-1}=x\left[\dfrac{1}{(1-x)^2}\right]$ [from part (a)] $=\dfrac{x}{(1-x)^2}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n}=\sum\limits_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\dfrac{1/2}{(1-1/2)^2}=2.$

(c) (i) $\sum\limits_{n=2}^{\infty}n(n-1)x^n=x^2\sum\limits_{n=2}^{\infty}n(n-1)x^{n-2}=x^2\dfrac{d}{dx}\left[\sum\limits_{n=1}^{\infty}nx^{n-1}\right]=x^2\dfrac{d}{dx}\dfrac{1}{(1-x)^2}$

$=x^2\dfrac{2}{(1-x)^3}=\dfrac{2x^2}{(1-x)^3}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty}\dfrac{n^2-n}{2^n}=\sum\limits_{n=2}^{\infty}n(n-1)\left(\frac{1}{2}\right)^n=\dfrac{2(1/2)^2}{(1-1/2)^3}=4.$

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n}=\sum\limits_{n=1}^{\infty}\dfrac{n^2-n}{2^n}+\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n}=4+2=6.$

39. By Example 7, $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have $\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{3}\right)^n \dfrac{1}{\sqrt{3}} \dfrac{1}{2n+1}$, so

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

40. (a) $\displaystyle\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \int_0^{1/2} \frac{dx}{(x - 1/2)^2 + 3/4}$ $\quad\left[x - \frac{1}{2} = \frac{\sqrt{3}}{2}u,\ u = \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right),\ dx = \frac{\sqrt{3}}{2}\,du\right]$

$$= \int_{-1/\sqrt{3}}^{0} \frac{(\sqrt{3}/2)\,du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3}\Big[\tan^{-1} u\Big]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}}\left[0 - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$$

(b) $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} \quad\Rightarrow$

$$\frac{1}{x^2-x+1} = (x+1)\left(\frac{1}{1+x^3}\right) = (x+1)\frac{1}{1-(-x^3)} = (x+1)\sum_{n=0}^{\infty}(-1)^n x^{3n}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{3n+1} + \sum_{n=0}^{\infty}(-1)^n x^{3n} \quad \text{for } |x| < 1 \quad\Rightarrow$$

$$\int \frac{dx}{x^2-x+1} = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1} \quad \text{for } |x| < 1 \quad\Rightarrow$$

$$\int_0^{1/2} \frac{dx}{x^2-x+1} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{4\cdot 8^n(3n+2)} + \frac{1}{2\cdot 8^n(3n+1)}\right] = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

By part (a), this equals $\dfrac{\pi}{3\sqrt{3}}$, so $\pi = \dfrac{3\sqrt{3}}{4}\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{8^n}\left(\dfrac{2}{3n+1} + \dfrac{1}{3n+2}\right)$.

11.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum\limits_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Equation 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.