

Determine whether the series is:

☐ absolutely convergent (AC)

☐ conditionally convergent (CC)

☐ divergent (D)

AC

1. $\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series [$|r| = \frac{1}{3} < 1$], so $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ converges by the Comparison Test.

also $\sum_{n=1}^{\infty} \frac{n}{n+2}$

D

3. $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{n+2} = 1$, so $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n \frac{n}{n+2}$ does not exist. Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ diverges by the Test for Divergence.

AC

5. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 2^n}{(-5)^{n+1}} \cdot \frac{(-5)^n}{n^2 2^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{2(n+1)^2}{5n^2} = \frac{2}{5} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = \frac{2}{5}(1) = \frac{2}{5} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$ converges by the Ratio Test.

D

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\int \frac{1}{x\sqrt{\ln x}} dx$ $\left[\begin{array}{l} u = \ln x, \\ du = dx/x \end{array} \right] = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t \rightarrow \infty} [2\sqrt{\ln x}]_2^t = \lim_{t \rightarrow \infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

AC

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get
- $$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| = \lim_{k \rightarrow \infty} \left[\left(\frac{k+1}{k} \right)^2 \cdot \frac{1}{e} \right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$$
- , so the series converges.

CC

11. $b_n = \frac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

$f(x) = \frac{1}{x \ln x}$ for $x \geq 2$ is continuous, positive, decreasing &
 $f(n) = a_n$ for all n . So can use integral test. $\rightarrow$

11. continued

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{\ln x} \cdot \frac{dx}{x} = \lim_{t \rightarrow \infty} \ln(\ln x) \Big|_{x=2}^{x=\infty}$$

$$= \left[\lim_{t \rightarrow \infty} \ln(\ln t) \right] - \ln(\ln 2) = \infty,$$

So $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test.

Since $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ conv. but $\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n \ln n} \right|$ divg,

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is cond. conv.

AC

13. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \lim_{n \rightarrow \infty} \frac{3(n+1)^2}{(n+1)n^2} = 3 \lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

AC

15. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n+2)[3(n+1)+2]} \cdot \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$,

so the series $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$ converges by the Ratio Test.

D

17. $\lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n \rightarrow \infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

$$\Rightarrow \sum |(-1)^n 2^{1/n}| = \sum 2^{1/n} \text{ also diverges}$$

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1/n}{1/(2\sqrt{n})} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the

Alternating Series Test.

Now need to consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$$

For $n \geq 3$

$$\frac{1}{n^{1/2}} \leq \frac{\ln n}{n^{1/2}}$$

$$\sum_{n=3}^{\infty} \frac{1}{n^{1/2}} = \infty \quad (\text{p-series, } p = \frac{1}{2} < 1) \quad \text{so}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}} = \infty \quad \text{by CT.}$$

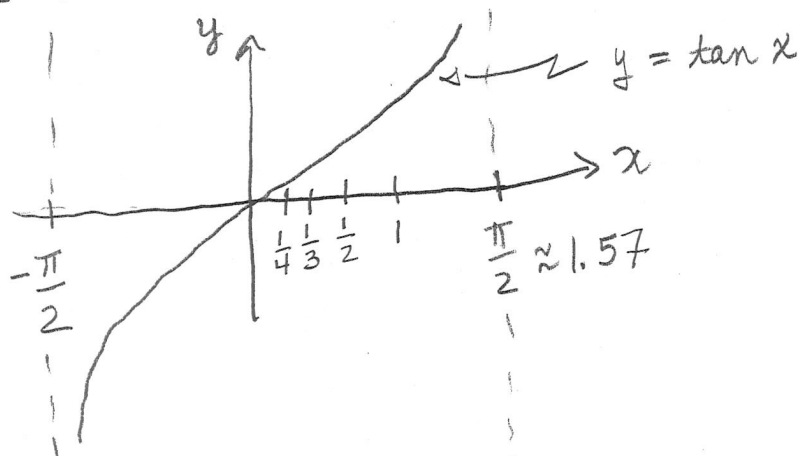
21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\sec^2(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \sec^2(1/x) = 1^2 = 1 > 0. \text{ Since}$$

$\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

Note: for $n=1, 2, 3, \dots$, $\tan \frac{1}{n} > 0$



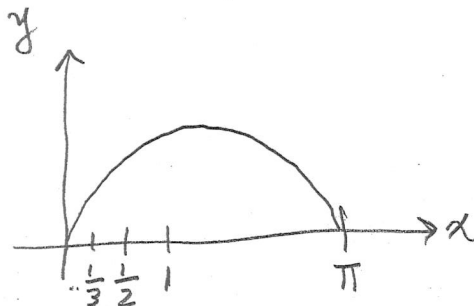
25. Use the Ratio Test. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n \rightarrow \infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

27. $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^t$ [using integration by parts] $\stackrel{H}{=} 1$. So $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.

31. $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{5^k}{3^k + 4^k} = [\text{divide by } 4^k] \lim_{k \rightarrow \infty} \frac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim_{k \rightarrow \infty} \left(\frac{3}{4}\right)^k = 0$ and $\lim_{k \rightarrow \infty} \left(\frac{5}{4}\right)^k = \infty$.
Thus, $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

33. Let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ [$p = 3/2 > 1$].

Note if n is a natural number ($n=1, 2, 3, \dots$) then $\sin(1/n) > 0$.



35. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{n^2/n} = \lim_{n \rightarrow \infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim_{n \rightarrow \infty} (1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$ converges by the Root Test.

37. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum_{n=1}^{\infty} (2^{1/n} - 1)^n$ converges by the Root Test.

Note if $n=1, 2, 3, \dots$ then

$$1 < 2 \Rightarrow 1^{1/n} < 2^{1/n} \Rightarrow 1 < 2^{1/n} \Rightarrow 0 < (2^{1/n} - 1)$$