

40. (a) The sum of the areas of the n rectangles in the graph to the right is

$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Now $\int_1^{n+1} \frac{dx}{x}$ is less than this sum because

the rectangles extend above the curve $y = 1/x$, so

$$\int_1^{n+1} \frac{1}{x} dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \text{ and since}$$

$$\ln n < \ln(n+1), 0 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = t_n.$$

- (b) The area under $f(x) = 1/x$ between $x = n$ and $x = n+1$ is

$$\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n, \text{ and this is clearly greater than the area of}$$

the inscribed rectangle in the figure to the right [which is $\frac{1}{n+1}$], so

$$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1} > 0, \text{ and so } t_n > t_{n+1}, \text{ so } \{t_n\} \text{ is a decreasing sequence.}$$

- (c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n . Thus, $0 < t_n \leq t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by the Monotonic Sequence Theorem.

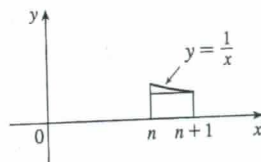
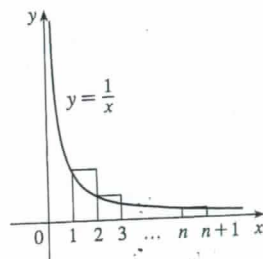
41. $b^{\ln n} = (e^{\ln b})^{\ln n} = (e^{\ln n})^{\ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}$. This is a p -series, which converges for all b such that $-\ln b > 1 \Leftrightarrow$

$$\ln b < -1 \Leftrightarrow b < e^{-1} \Leftrightarrow b < 1/e \text{ [with } b > 0].$$

42. For the series $\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$,

$$\begin{aligned} s_n &= \sum_{i=1}^n \left(\frac{c}{i} - \frac{1}{i+1} \right) = \left(\frac{c}{1} - \frac{1}{2} \right) + \left(\frac{c}{2} - \frac{1}{3} \right) + \left(\frac{c}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{c}{n} - \frac{1}{n+1} \right) \\ &= \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \frac{c-1}{4} + \cdots + \frac{c-1}{n} - \frac{1}{n+1} = c + (c-1) \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \right) - \frac{1}{n+1} \end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[c + (c-1) \sum_{i=2}^n \frac{1}{i} - \frac{1}{n+1} \right]$. Since a constant multiple of a divergent series is divergent, the last limit exists only if $c-1 = 0$, so the original series converges only if $c = 1$.



11.4 The Comparison Tests

- (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)
- (b) If $a_n < b_n$ for all n , then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]
- (a) If $a_n > b_n$ for all n , then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]
- (b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\frac{n}{2n^3+1} < \frac{n}{2n^3} = \frac{1}{2n^2} < \frac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p -series with $p = 2 > 1$.
4. $\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges because it is a p -series with $p = 1 \leq 1$ (the harmonic series).
5. $\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges because it is a p -series with $p = \frac{1}{2} \leq 1$.
6. $\frac{n-1}{n^2\sqrt{n}} < \frac{n}{n^2n^{1/2}} = \frac{1}{n^{3/2}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges because it is a p -series with $p = \frac{3}{2} > 1$.
7. $\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series ($|r| = \frac{9}{10} < 1$), so $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges by the Comparison Test.
8. $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.
9. $\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ converges by comparison with the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ [$p = 2 > 1$].
10. $\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3n^2}$. $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p -series [$p = 2 > 1$]. The terms of the given series are positive for $n > 1$, which is good enough.
11. $\frac{n-1}{n4^n}$ is positive for $n > 1$ and $\frac{n-1}{n4^n} < \frac{n}{n4^n} = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$, so $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$.
12. $\frac{1+\sin n}{10^n} \leq \frac{2}{10^n}$ and $\sum_{n=0}^{\infty} \frac{2}{10^n} = 2 \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.
13. $\frac{\arctan n}{n^{1.2}} < \frac{\pi/2}{n^{1.2}}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$ converges by comparison with $\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$, which converges because it is a constant times a p -series with $p = 1.2 > 1$.
14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p -series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ [$p = \frac{1}{2} \leq 1$].

15. $\frac{2+(-1)^n}{n\sqrt{n}} \leq \frac{3}{n\sqrt{n}}$, and $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$.

$[p = \frac{3}{2} > 1]$, so the given series converges by the Comparison Test.

16. $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad [p = \frac{3}{2} > 1].$$

17. Use the Limit Comparison Test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0. \text{ Since the harmonic series } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, so does}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}.$$

18. Use the Limit Comparison Test with $a_n = \frac{1}{2n+3}$ and $b_n = \frac{1}{n}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{2n+3} = \lim_{n \rightarrow \infty} \frac{1}{2+(3/n)} = \frac{1}{2} > 0$.

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2n+3}$.

19. Use the Limit Comparison Test with $a_n = \frac{1+4^n}{1+3^n}$ and $b_n = \frac{4^n}{3^n}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1+4^n}{\frac{1+3^n}{\frac{4^n}{3^n}}} = \lim_{n \rightarrow \infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n \rightarrow \infty} \frac{1+4^n}{4^n} \cdot \frac{3^n}{1+3^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{4^n} + 1 \right) \cdot \frac{1}{\frac{1}{3^n} + 1} = 1 > 0$$

Since the geometric series $\sum b_n = \sum \left(\frac{4}{3}\right)^n$ diverges, so does $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$. Alternatively, use the Comparison Test with

$$\frac{1+4^n}{1+3^n} > \frac{1+4^n}{3^n+3^n} > \frac{4^n}{2(3^n)} = \frac{1}{2} \left(\frac{4}{3}\right)^n \text{ or use the Test for Divergence.}$$

20. $4^n > n$ for all $n \geq 1$ since the function $f(x) = 4^x - x$ satisfies $f(1) = 3$ and $f'(x) = 4^x \ln 4 - 1 > 0$ for $x \geq 1$, so $\frac{n+4^n}{n+6^n} < \frac{4^n+4^n}{n+6^n} < \frac{2 \cdot 4^n}{6^n} = 2\left(\frac{4}{6}\right)^n$, so the series $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ converges by comparison with $2 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$, which is a constant multiple of a convergent geometric series $[|r| = \frac{2}{3} < 1]$.

Or: Use the Limit Comparison Test with $a_n = \frac{n+4^n}{n+6^n}$ and $b_n = \left(\frac{2}{3}\right)^n$.

21. Use the Limit Comparison Test with $a_n = \frac{\sqrt{n+2}}{2n^2+n+1}$ and $b_n = \frac{1}{n^{3/2}}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{3/2} \sqrt{n+2}}{2n^2+n+1} = \lim_{n \rightarrow \infty} \frac{(n^{3/2} \sqrt{n+2}) / (n^{3/2} \sqrt{n})}{(2n^2+n+1)/n^2} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+2/n}}{2+1/n+1/n^2} = \frac{\sqrt{1}}{2} = \frac{1}{2} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p -series $[p = \frac{3}{2} > 1]$, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ also converges.

22. Use the Limit Comparison Test with $a_n = \frac{n+2}{(n+1)^3}$ and $b_n = \frac{1}{n^2}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n}}{\left(1 + \frac{1}{n}\right)^3} = 1 > 0. \text{ Since } \sum_{n=3}^{\infty} \frac{1}{n^2} \text{ is a convergent (partial) } p\text{-series } [p = 2 > 1],$$

the series $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ also converges.

23. Use the Limit Comparison Test with $a_n = \frac{5+2n}{(1+n^2)^2}$ and $b_n = \frac{1}{n^3}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n \rightarrow \infty} \frac{5n^3 + 2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n} + 2}{\left(\frac{1}{n^2} + 1\right)^2} = 2 > 0. \text{ Since } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ is a convergent}$$

p -series [$p = 3 > 1$], the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \frac{n^2 - 5n}{n^3 + n + 1}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 - 5n^2}{n^3 + n + 1} = \lim_{n \rightarrow \infty} \frac{1 - 5/n}{1 + 1/n^2 + 1/n^3} = 1 > 0$,

so $\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n \rightarrow \infty} \frac{1/n^2 + 1/n + 1}{\sqrt{1/n^6 + 1/n^4 + 1}} = 1 > 0$,

so $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

26. If $a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{7/3} + 5n^{4/3}}{(n^7 + n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n \rightarrow \infty} \frac{1 + 5/n}{[(n^7 + n^2)/n^7]^{1/3}} = \lim_{n \rightarrow \infty} \frac{1 + 5/n}{(1 + 1/n^5)^{1/3}} = \frac{1 + 0}{(1 + 0)^{1/3}} = 1 > 0,$$

so $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p -series $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$.

27. Use the Limit Comparison Test with $a_n = \left(1 + \frac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 1 > 0$. Since

$\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series [$|r| = \frac{1}{e} < 1$], the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$ also converges.

28. $\frac{e^{1/n}}{n} > \frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

29. Clearly $n! = n(n-1)(n-2) \cdots (3)(2) \geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 = 2^{n-1}$, so $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent geometric

series [$|r| = \frac{1}{2} < 1$], so $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the Comparison Test.

30. $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)n}{n \cdot n \cdot n \cdots n \cdot n} \leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges [$p = 2 > 1$], $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 > 0. \text{ Since } \sum_{n=1}^{\infty} b_n \text{ is the divergent harmonic series,}$$

$\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\cos(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1.]$$

32. Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{n^{1+1/n}} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = 1$

[since $\lim_{x \rightarrow \infty} x^{1/x} = 1$ by l'Hospital's Rule], so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [harmonic series] $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

33. $\sum_{n=1}^{10} \frac{1}{\sqrt{n^4+1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{17}} + \frac{1}{\sqrt{82}} + \cdots + \frac{1}{\sqrt{10,001}} \approx 1.24856$. Now $\frac{1}{\sqrt{n^4+1}} < \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}$, so the error is

$$R_{10} \leq T_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{x}\right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10} = 0.1.$$

34. $\sum_{n=1}^{10} \frac{\sin^2 n}{n^3} = \frac{\sin^2 1}{1} + \frac{\sin^2 2}{8} + \frac{\sin^2 3}{27} + \cdots + \frac{\sin^2 10}{1000} \approx 0.83253$. Now $\frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$, so the error is

$$R_{10} \leq T_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2x^2}\right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{200}\right) = \frac{1}{200} = 0.005.$$

35. $\sum_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is

$$R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2} \text{ [geometric series]} \approx 0.00098.$$

36. $\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649,539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n \cdot 3^n} = \frac{1}{3^n}$, so the error is

$$R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1-1/3} \approx 0.0000085.$$

37. Since $\frac{d_n}{10^n} \leq \frac{9}{10^n}$ for each n , and since $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3 \cdots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim_{n \rightarrow \infty} \frac{1}{n^p \ln n} = \infty$. If $0 \leq p \leq 1$, then $n^p \ln n \leq n \ln n \Rightarrow$

$\frac{1}{n^p \ln n} \geq \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 11.3.21), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison

Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges.

(Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

39. Since $\sum a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \leq a_n < 1$ for

all $n > N \Rightarrow 0 \leq a_n^2 \leq a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim_{n \rightarrow \infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) (i) If $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{1}{n^2}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$, so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by

part (a).

(ii) If $a_n = \frac{\ln n}{\sqrt{ne^n}}$ and $b_n = \frac{1}{e^n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$. Now

$\sum b_n$ is a convergent geometric series with ratio $r = 1/e$ ($|r| < 1$), so $\sum a_n$ converges by part (a).

41. (a) Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 11.1.5.)

Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \geq 2$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{x \rightarrow \infty} \frac{x}{\ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{1/x} = \lim_{x \rightarrow \infty} x = \infty$,

so by part (a), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \ln n = \lim_{x \rightarrow \infty} \ln x = \infty$,

so $\sum_{n=1}^{\infty} a_n$ diverges by part (a).

42. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} na_n > 0$ we know that either both

series converge or both series diverge, and we also know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [p -series with $p = 1$]. Therefore, $\sum a_n$ must be

divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$ by

Theorem 11.2.6. Therefore, $\lim_{n \rightarrow \infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$.

Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim_{n \rightarrow \infty} a_n = 0$ by Theorem 11.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.3.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.
46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \rightarrow \infty$, so for some integer N , $a_n \leq 1$ for all $n \geq N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \leq \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

11.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.
- (b) An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if $0 < b_{n+1} \leq b_n$ for all n and $\lim_{n \rightarrow \infty} b_n = 0$. (This is the Alternating Series Test.)
- (c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1} ; that is, $|R_n| \leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)
2. $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$. Here $a_n = (-1)^n \frac{n}{n+2}$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.
3. $\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{n+6}$. Now $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series converges by the Alternating Series Test.
4. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}}$. Now $b_n = \frac{1}{\sqrt{n+1}} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series converges by the Alternating Series Test.
5. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Now $b_n = \frac{1}{2n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series converges by the Alternating Series Test.
6. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Now $b_n = \frac{1}{\ln(n+4)} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n \rightarrow \infty} b_n = 0$, so the series converges by the Alternating Series Test.
7. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3-1/n}{2+1/n} = \frac{3}{2} \neq 0$. Since $\lim_{n \rightarrow \infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.