11.4 CT / LCT ~~Alt~~ n

11.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

   (b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

   (b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\dfrac{n}{2n^3+1} < \dfrac{n}{2n^3} = \dfrac{1}{2n^2} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n}{2n^3+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

5. $\dfrac{n+1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n\sqrt{n}}$ diverges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$.

7. $\dfrac{9^n}{3+10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{9^n}{3+10^n}$ converges by the Comparison Test.

9. $\dfrac{\cos^2 n}{n^2+1} \leq \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2+1}$ converges by comparison with the p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $[p = 2 > 1]$.

11. $\dfrac{n-1}{n4^n}$ is positive for $n > 1$ and $\dfrac{n-1}{n4^n} < \dfrac{n}{n4^n} = \dfrac{1}{4^n} = \left(\dfrac{1}{4}\right)^n$, so $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{4}\right)^n$.

13. $\dfrac{\arctan n}{n^{1.2}} < \dfrac{\pi/2}{n^{1.2}}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{\arctan n}{n^{1.2}}$ converges by comparison with $\dfrac{\pi}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1.2}}$, which converges because it is a constant times a p-series with $p = 1.2 > 1$.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0.$$ Since the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

21. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+2}}{2n^2+n+1}$ and $b_n = \dfrac{1}{n^{3/2}}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+2}}{2n^2+n+1} = \lim_{n\to\infty} \frac{(n^{3/2}\sqrt{n+2})/(n^{3/2}\sqrt{n})}{(2n^2+n+1)/n^2} = \lim_{n\to\infty} \frac{\sqrt{1+2/n}}{2+1/n+1/n^2} = \frac{\sqrt{1}}{2} = \frac{1}{2} > 0.$$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is a convergent p-series $\left[p = \frac{3}{2} > 1\right]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n+2}}{2n^2+n+1}$ also converges.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is a convergent geometric series $\left[|r| = \frac{1}{2} < 1\right]$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!}$ converges by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum\limits_{n=1}^{\infty} \sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1.]$$

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N$ $\Rightarrow$ $0 \leq a_n < 1$ for all $n > N$ $\Rightarrow$ $0 \leq a_n^2 \leq a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim\limits_{n\to\infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) (i) If $a_n = \dfrac{\ln n}{n^3}$ and $b_n = \dfrac{1}{n^2}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = \dfrac{\ln n}{\sqrt{n}e^n}$ and $b_n = \dfrac{1}{e^n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1/(2\sqrt{x})} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$. Now $\sum b_n$ is a convergent geometric series with ratio $r = 1/e$ $[|r| < 1]$, so $\sum a_n$ converges by part (a).

41. (a) Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, there is an integer N such that $\dfrac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 11.1.5.) Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \dfrac{1}{\ln n}$ and $b_n = \dfrac{1}{n}$ for $n \geq 2$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{\ln n} = \lim\limits_{x\to\infty} \dfrac{x}{\ln x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1}{1/x} = \lim\limits_{x\to\infty} x = \infty$, so by part (a), $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ is divergent.

(ii) If $a_n = \dfrac{\ln n}{n}$ and $b_n = \dfrac{1}{n}$, then $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \ln n = \lim\limits_{x\to\infty} \ln x = \infty$, so $\sum\limits_{n=1}^{\infty} a_n$ diverges by part (a).

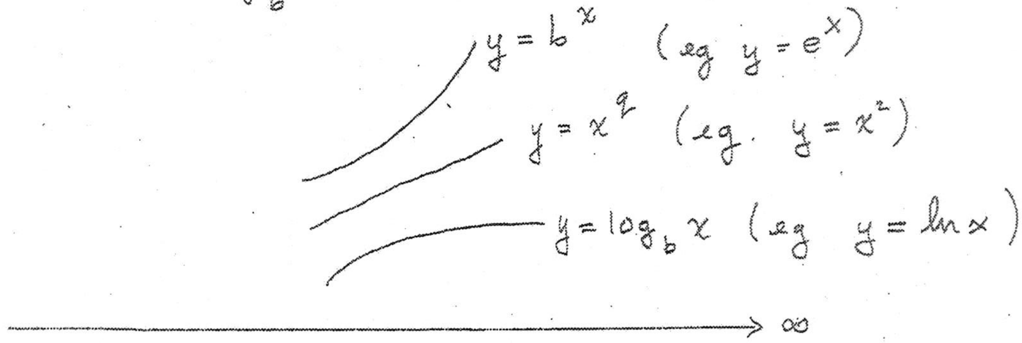
$\to$ More on 40b & 41b in a few pages.

43. $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n\to\infty} na_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [p-series with $p = 1$]. Therefore, $\sum a_n$ must be divergent.

Helpful Intuition

For any power $0 < q < \infty$ and any base $b > 1$,
for n large enough

$$\log_b n \le n^q \le b^n.$$



In fact (L'Hopital's Rule)

$$\lim_{n\to\infty} \frac{\log_b n}{n^q} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n^q}{b^n} = 0.$$

So to figure out what happens to a series which involves a $\log_b n$ or b^n, remember

- $\log_b n$ grows "super slow" compared to n^q
- b^n grows "super fast" compared to n^q

4Db via Helpful Intuition

(i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ ☒ conv. ☐ dvg.

Let $q = 1.5$

$$0 \le \frac{\ln n}{n^3} \le \frac{n^q}{n^3} = \frac{1}{n^{3-q}} = \frac{1}{n^{3-1.5}} = \frac{1}{n^{1.5}}$$

Helpful Intuition for any $0<q<\infty$ & n big enuf.

pick any q with
- $0<q<\infty$
- $3-q>1 \overset{\text{i.e.}}{\Longleftrightarrow} \underset{2}{3-1} > q$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ conv (p-series, $p=1.5>1$)

so by CT $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ conv.

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ ☒ conv ☐ dvg

$$0 \le \frac{\ln n}{\sqrt{n}\, e^n} \le \frac{n^{q_1}}{n^{1/2}\, n^{q_2}} = \frac{1}{n^{1/2+q_2-q_1}} = \frac{1}{n^{1/2+17-16}} = \frac{1}{n^{3/2}}$$

$\boxed{\text{Helpful Intuition, for any } 0<q_1, q_2<\infty \text{ and } n \text{ big enuf}}$

pick q_1 & q_2 with
- $0<q_1<\infty$
- $0<q_2<\infty$
- $\frac{1}{2}+q_2-q_1>1$ $\Updownarrow$ $q_2-q_1>\frac{1}{2}$

so $\boxed{q_2 = 17,\ q_1 = 16}$

$\sum \frac{1}{n^{3/2}}$ conv. (p-series, $p=\frac{3}{2}>1$)

so by CT

$\sum \frac{\ln n}{\sqrt{n}\, e^n}$ conv

41b. via Helpful Intuition

(i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ ☐ conv ☒ dvg

Remark: $\frac{1}{\ln n} \overset{n=1}{=\!=} \frac{1}{\ln 1} = \frac{1}{0} \ldots$ no can do

$$\frac{1}{\ln n} \geq \frac{1}{n^q} \overset{q=1/2}{=\!=} \frac{1}{n^{1/2}}$$

for any $0 < q < \infty$ & n big enough

pick q so that
- $0 < q < \infty$
- $q < 1$

so $q = \frac{1}{2}$ works

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ dvg (p-series, $p = \frac{1}{2} < 1$) so $\sum_{n=2}^{\infty} \ln n$ dvg by C.T.

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ cannot be done w/ helpful intuition.

The only way we could use the helpful intuition is like:

$$0 \leq \frac{\ln n}{n} \leq \frac{n^q}{n} \overset{\text{algebra}}{=\!=} \frac{1}{n^{1-q}}$$

for any $0<q<\infty$ & n big enuf

$0<q<\infty \Rightarrow -\infty < -q < 0$

$\Downarrow$

$-\infty < 1-q < 1 \Rightarrow \sum \frac{1}{n^{1-q}}$ dvg p-series $p = 1-q$, $p < 1$

So w/ helpful intuition we can bound $\sum \frac{\ln n}{n}$ above by a divergent series but this doesn't help.