§ 11.2 Challenging but important

65, 66, 68, 69, 70, 73a.

65. The series $1-1+1-1+1-1+\cdots$ diverges (geometric series with $r=-1$) so we cannot say that $0=1-1+1-1+1-1+\cdots$.

66. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \frac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent by the Test for Divergence.

67. $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} \sum_{i=1}^{n} ca_i = \lim_{n\to\infty} c\sum_{i=1}^{n} a_i = c\lim_{n\to\infty} \sum_{i=1}^{n} a_i = c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

68. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

69. Suppose on the contrary that $\sum(a_n+b_n)$ converges. Then $\sum(a_n+b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum[(a_n+b_n)-a_n]$ would also be convergent. But $\sum[(a_n+b_n)-a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

70. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n+b_n) = \sum 0$, which converges with sum 0.

71. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \le 1000$ for all n. So by the Monotonic Sequence Theorem, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

73. (a) At the first step, only the interval $\left(\frac13,\frac23\right)$ (length $\frac13$) is removed. At the second step, we remove the intervals $\left(\frac19,\frac29\right)$ and $\left(\frac79,\frac89\right)$, which have a total length of $2\cdot\left(\frac13\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac13\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac13\right)^n$, for a length of $2^{n-1}\cdot\left(\frac13\right)^n = \frac13\left(\frac23\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty}\frac13\left(\frac23\right)^{n-1} = \frac{1/3}{1-2/3} = 1$ [geometric series with $a=\frac13$ and $r=\frac23$]. Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac13\right)^n, \left(\frac23\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1-\left(\frac23\right)^n, 1-\left(\frac13\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac13, \frac23, \frac19, \frac29, \frac79$, and $\frac89$.

(b) The area removed at the first step is $\frac19$; at the second step, $8\cdot\left(\frac19\right)^2$; at the third step, $(8)^2\cdot\left(\frac19\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac19\right)^n = \frac19\left(\frac89\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty}\frac19\left(\frac89\right)^{n-1} = \frac{1/9}{1-8/9} = 1.$$