§ 11.2 Sol'n to the "Basics" homework problems.

11.2 Series

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

   (b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

10. (a) Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

    (b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

19. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \frac{\pi}{3}$. Since $|r| > 1$, the series diverges.

21. $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since each of its partial sums is $\frac{1}{2}$ times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges. $\left[\text{If } \sum_{n=1}^{\infty}\frac{1}{2n} \text{ were to converge, then } \sum_{n=1}^{\infty}\frac{1}{n} \text{ would also have to converge by Theorem 8(i).}\right]$
    In general, constant multiples of divergent series are divergent.

23. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{k^2}{k^2-1} = 1 \neq 0$.

25. Converges.

$$\sum_{n=1}^{\infty}\frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}+\frac{2^n}{3^n}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right] \quad \text{[sum of two convergent geometric series]}$$
$$= \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = \frac{1}{2} + 2 = \frac{5}{2}$$

29. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverges by the Test for Divergence since
$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right) = \ln\frac{1}{2} \neq 0$.

31. $\sum_{n=1}^{\infty} \arctan n$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\arctan n = \frac{\pi}{2} \neq 0$.

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35. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right)$$
$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

38. For the series $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$,

$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + [\ln n - \ln(n+1)] = \ln 1 - \ln(n+1) = -\ln(n+1)$

[telescoping series]

Thus, $\lim_{n\to\infty} s_n = -\infty$, so the series is divergent.

But of course you found the sum of the convergent geometric series using our method from class instead of plugging into some formula that you might forget in 2 years when you need it.