

## 10.4 Areas and Lengths in Polar Coordinates

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1.  $r = \theta^2$ ,  $0 \leq \theta \leq \frac{\pi}{4}$ .  $A = \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = \int_0^{\pi/4} \frac{1}{2} (\theta^2)^2 d\theta = \int_0^{\pi/4} \frac{1}{2} \theta^4 d\theta = \left[ \frac{1}{10} \theta^5 \right]_0^{\pi/4} = \frac{1}{10} \left( \frac{\pi}{4} \right)^5 = \frac{1}{10240} \pi^5$

2.  $r = e^{\theta/2}$ ,  $\pi \leq \theta \leq 2\pi$ .  $A = \int_{\pi}^{2\pi} \frac{1}{2} (e^{\theta/2})^2 d\theta = \int_{\pi}^{2\pi} \frac{1}{2} e^\theta d\theta = \frac{1}{2} \left[ e^\theta \right]_{\pi}^{2\pi} = \frac{1}{2} (e^{2\pi} - e^\pi)$

3.  $r = \sin \theta$ ,  $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$ .

$$\begin{aligned} A &= \int_{\pi/3}^{2\pi/3} \frac{1}{2} \sin^2 \theta d\theta = \frac{1}{4} \int_{\pi/3}^{2\pi/3} (1 - \cos 2\theta) d\theta = \frac{1}{4} \left[ \theta - \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{2\pi/3} = \frac{1}{4} \left[ \frac{2\pi}{3} - \frac{1}{2} \sin \frac{4\pi}{3} - \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} \right] \\ &= \frac{1}{4} \left[ \frac{2\pi}{3} - \frac{1}{2} \left( -\frac{\sqrt{3}}{2} \right) - \frac{\pi}{3} + \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) \right] = \frac{1}{4} \left( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8} \end{aligned}$$

4.  $r = \sqrt{\sin \theta}$ ,  $0 \leq \theta \leq \pi$ .  $A = \int_0^{\pi} \frac{1}{2} (\sqrt{\sin \theta})^2 d\theta = \int_0^{\pi} \frac{1}{2} \sin \theta d\theta = \left[ -\frac{1}{2} \cos \theta \right]_0^{\pi} = \frac{1}{2} + \frac{1}{2} = 1$

5.  $r = \sqrt{\theta}$ ,  $0 \leq \theta \leq 2\pi$ .  $A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (\sqrt{\theta})^2 d\theta = \int_0^{2\pi} \frac{1}{2} \theta d\theta = \left[ \frac{1}{4} \theta^2 \right]_0^{2\pi} = \pi^2$

6.  $r = 1 + \cos \theta$ ,  $0 \leq \theta \leq \pi$ .

$$\begin{aligned} A &= \int_0^{\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \frac{1}{2} \int_0^{\pi} [1 + 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta \\ &= \frac{1}{2} \int_0^{\pi} \left( \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2} \left[ \frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi} = \frac{1}{2} \left( \frac{3}{2}\pi + 0 + 0 \right) - \frac{1}{2}(0) = \frac{3\pi}{4} \end{aligned}$$

7.  $r = 4 + 3 \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}((4 + 3 \sin \theta)^2) d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 24 \sin \theta + 9 \sin^2 \theta) d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 9 \sin^2 \theta) d\theta \quad [\text{by Theorem 5.5.7(b)}] \\ &= \frac{1}{2} \cdot 2 \int_0^{\pi/2} [16 + 9 + \frac{1}{2}(1 - \cos 2\theta)] d\theta \quad [\text{by Theorem 5.5.7(a)}] \\ &= \int_0^{\pi/2} \left(\frac{41}{2} - \frac{9}{2} \cos 2\theta\right) d\theta = \left[\frac{41}{2}\theta - \frac{9}{4} \sin 2\theta\right]_0^{\pi/2} = \left(\frac{41\pi}{4} - 0\right) - (0 - 0) = \frac{41\pi}{4} \end{aligned}$$

8.  $r = \sin 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$ .

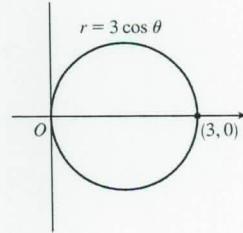
$$A = \int_0^{\pi/2} \frac{1}{2} \sin^2 2\theta d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{1}{4} [\theta - \frac{1}{4} \sin 4\theta]_0^{\pi/2} = \frac{1}{4} \left(\frac{\pi}{2}\right) = \frac{\pi}{8}$$

9. The area above the polar axis is bounded by  $r = 3 \cos \theta$  for  $\theta = 0$

to  $\theta = \pi/2$  [not  $\pi$ ]. By symmetry,

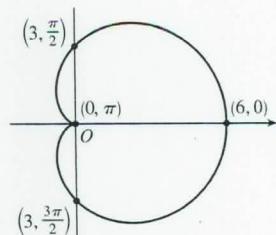
$$\begin{aligned} A &= 2 \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} (3 \cos \theta)^2 d\theta = 3^2 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 9 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{9}{2} [\theta + \frac{1}{2} \sin 2\theta]_0^{\pi/2} = \frac{9}{2} [(\frac{\pi}{2} + 0) - (0 + 0)] = \frac{9\pi}{4} \end{aligned}$$

Also, note that this is a circle with radius  $\frac{3}{2}$ , so its area is  $\pi (\frac{3}{2})^2 = \frac{9\pi}{4}$ .



10.  $A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} [3(1 + \cos \theta)]^2 d\theta$

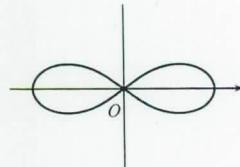
$$\begin{aligned} &= \frac{9}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{9}{2} \int_0^{2\pi} [1 + 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta \\ &= \frac{9}{2} [\frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta]_0^{2\pi} = \frac{27}{2}\pi \end{aligned}$$



11. The curve goes through the pole when  $\theta = \pi/4$ , so we'll find the area for

$0 \leq \theta \leq \pi/4$  and multiply it by 4.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 2 \int_0^{\pi/4} (4 \cos 2\theta) d\theta \\ &= 8 \int_0^{\pi/4} \cos 2\theta d\theta = 4 [\sin 2\theta]_0^{\pi/4} = 4 \end{aligned}$$

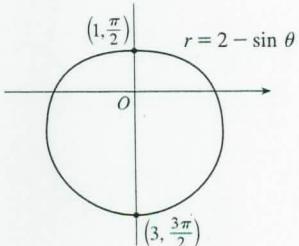


12. To find the area that the curve encloses, we'll double the area to the left of the vertical axis.

$$\begin{aligned} A &= 2 \int_{\pi/2}^{3\pi/2} \frac{1}{2} (2 - \sin \theta)^2 d\theta = \int_{\pi/2}^{3\pi/2} (4 - 4 \sin \theta + \sin^2 \theta) d\theta \\ &= \int_{\pi/2}^{3\pi/2} [4 - 4 \sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta = \int_{\pi/2}^{3\pi/2} (\frac{9}{2} - 4 \sin \theta - \frac{1}{2} \cos 2\theta) d\theta \\ &= [\frac{9}{2}\theta + 4 \cos \theta - \frac{1}{4} \sin 2\theta]_{\pi/2}^{3\pi/2} = (\frac{27\pi}{4}) - (\frac{9\pi}{4}) = \frac{9\pi}{2} \end{aligned}$$

Or: We could have doubled the area to the right of the vertical axis and integrated from  $-\pi/2$  to  $\pi/2$ .

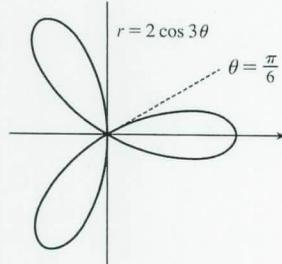
Or: We could have integrated from 0 to  $2\pi$  [simpler arithmetic].



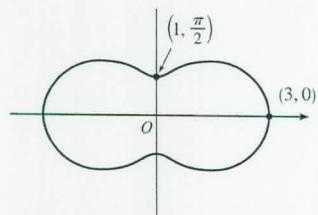
13. One-sixth of the area lies above the polar axis and is bounded by the curve

$$r = 2 \cos 3\theta \text{ for } \theta = 0 \text{ to } \theta = \pi/6.$$

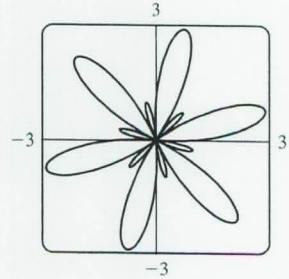
$$\begin{aligned} A &= 6 \int_0^{\pi/6} \frac{1}{2}(2 \cos 3\theta)^2 d\theta = 12 \int_0^{\pi/6} \cos^2 3\theta d\theta \\ &= \frac{12}{2} \int_0^{\pi/6} (1 + \cos 6\theta) d\theta \\ &= 6[\theta + \frac{1}{6} \sin 6\theta]_0^{\pi/6} = 6(\frac{\pi}{6}) = \pi \end{aligned}$$



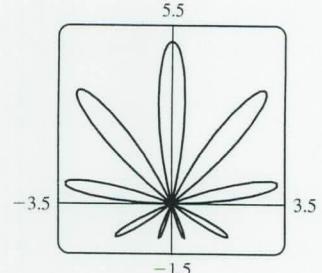
$$\begin{aligned} 14. A &= \int_0^{2\pi} \frac{1}{2}(2 + \cos 2\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos 2\theta + \cos^2 2\theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta) d\theta \\ &= \frac{1}{2} [\frac{9}{2}\theta + 2 \sin 2\theta + \frac{1}{8} \sin 4\theta]_0^{2\pi} = \frac{1}{2}(9\pi) = \frac{9\pi}{2} \end{aligned}$$



$$\begin{aligned} 15. A &= \int_0^{2\pi} \frac{1}{2}(1 + 2 \sin 6\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 4 \sin 6\theta + 4 \sin^2 6\theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [1 + 4 \sin 6\theta + 4 \cdot \frac{1}{2}(1 - \cos 12\theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (3 + 4 \sin 6\theta - 2 \cos 12\theta) d\theta \\ &= \frac{1}{2} [3\theta - \frac{2}{3} \cos 6\theta - \frac{1}{6} \sin 12\theta]_0^{2\pi} \\ &= \frac{1}{2} [(6\pi - \frac{2}{3} - 0) - (0 - \frac{2}{3} - 0)] = 3\pi \end{aligned}$$

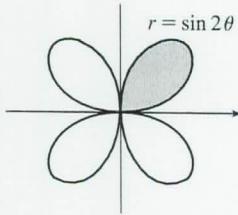


$$\begin{aligned} 16. A &= \int_0^{\pi} \frac{1}{2}(2 \sin \theta + 3 \sin 9\theta)^2 d\theta = 2 \int_0^{\pi/2} \frac{1}{2}(2 \sin \theta + 3 \sin 9\theta)^2 d\theta \\ &= \int_0^{\pi/2} (4 \sin^2 \theta + 12 \sin \theta \sin 9\theta + 9 \sin^2 9\theta) d\theta \\ &= \int_0^{\pi/2} [2(1 - \cos 2\theta) + 12 \cdot \frac{1}{2}(\cos(\theta - 9\theta) - \cos(\theta + 9\theta)) + \frac{9}{2}(1 - \cos 18\theta)] d\theta \\ &\quad [\text{integration by parts could be used for } \int \sin \theta \sin 9\theta d\theta] \\ &= \int_0^{\pi/2} (2 - 2 \cos 2\theta + 6 \cos 8\theta - 6 \cos 10\theta + \frac{9}{2} - \frac{9}{2} \cos 18\theta) d\theta \\ &= [\frac{13}{2}\theta - \sin 2\theta + \frac{3}{4} \sin 8\theta - \frac{3}{5} \sin 10\theta - \frac{1}{4} \sin 18\theta]_0^{\pi/2} = \frac{13\pi}{4} \end{aligned}$$

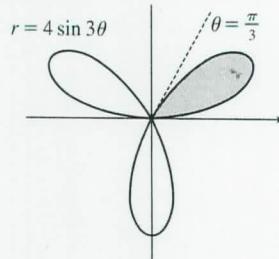


17. The shaded loop is traced out from  $\theta = 0$  to  $\theta = \pi/2$ .

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2}r^2 d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{1}{4} [\theta - \frac{1}{4} \sin 4\theta]_0^{\pi/2} \\ &= \frac{1}{4} (\frac{\pi}{2}) = \frac{\pi}{8} \end{aligned}$$



$$\begin{aligned} 18. A &= \int_0^{\pi/3} \frac{1}{2}(4 \sin 3\theta)^2 d\theta = 8 \int_0^{\pi/3} \sin^2 3\theta d\theta \\ &= 4 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta \\ &= 4 [\theta - \frac{1}{6} \sin 6\theta]_0^{\pi/3} = \frac{4\pi}{3} \end{aligned}$$



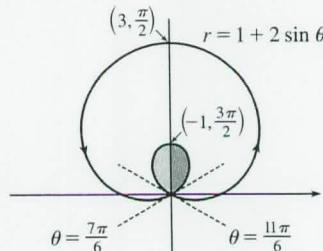
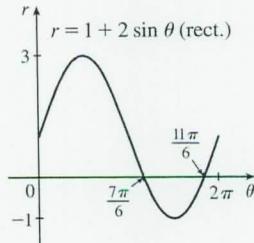
19.  $r = 0 \Rightarrow 3 \cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{10}$ .

$$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2}(3 \cos 5\theta)^2 d\theta = \int_0^{\pi/10} 9 \cos^2 5\theta d\theta = \frac{9}{2} \int_0^{\pi/10} (1 + \cos 10\theta) d\theta = \frac{9}{2} [\theta + \frac{1}{10} \sin 10\theta]_0^{\pi/10} = \frac{9\pi}{20}$$

20.  $r = 0 \Rightarrow 2 \sin 6\theta = 0 \Rightarrow 6\theta = 0 \text{ or } \pi \Rightarrow \theta = 0 \text{ or } \frac{\pi}{6}$ .

$$A = \int_0^{\pi/6} \frac{1}{2}(2 \sin 6\theta)^2 d\theta = \int_0^{\pi/6} 2 \sin^2 6\theta d\theta = 2 \int_0^{\pi/6} \frac{1}{2}(1 - \cos 12\theta) d\theta = [\theta - \frac{1}{12} \sin 12\theta]_0^{\pi/6} = \frac{\pi}{6}$$

21.



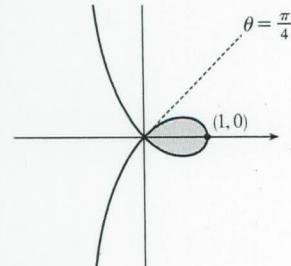
This is a limacon, with inner loop traced out between  $\theta = \frac{7\pi}{6}$  and  $\frac{11\pi}{6}$  [found by solving  $r = 0$ ].

$$\begin{aligned} A &= 2 \int_{7\pi/6}^{3\pi/2} \frac{1}{2}(1 + 2 \sin \theta)^2 d\theta = \int_{7\pi/6}^{3\pi/2} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta = \int_{7\pi/6}^{3\pi/2} [1 + 4 \sin \theta + 4 \cdot \frac{1}{2}(1 - \cos 2\theta)] d\theta \\ &= [\theta - 4 \cos \theta + 2\theta - \sin 2\theta]_{7\pi/6}^{3\pi/2} = \left(\frac{9\pi}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

22. To determine when the strophoid  $r = 2 \cos \theta - \sec \theta$  passes through the pole, we solve

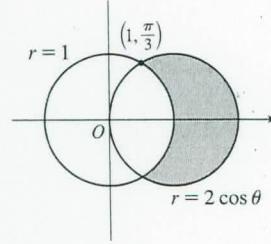
$$\begin{aligned} r = 0 &\Rightarrow 2 \cos \theta - \frac{1}{\cos \theta} = 0 \Rightarrow 2 \cos^2 \theta - 1 = 0 \Rightarrow \cos^2 \theta = \frac{1}{2} \Rightarrow \\ \cos \theta &= \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4} \text{ for } 0 \leq \theta \leq \pi \text{ with } \theta \neq \frac{\pi}{2}. \end{aligned}$$

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \frac{1}{2}(2 \cos \theta - \sec \theta)^2 d\theta = \int_0^{\pi/4} (4 \cos^2 \theta - 4 + \sec^2 \theta) d\theta \\ &= \int_0^{\pi/4} [4 \cdot \frac{1}{2}(1 + \cos 2\theta) - 4 + \sec^2 \theta] d\theta = \int_0^{\pi/4} (-2 + 2 \cos 2\theta + \sec^2 \theta) d\theta \\ &= [-2\theta + \sin 2\theta + \tan \theta]_0^{\pi/4} = \left(-\frac{\pi}{2} + 1 + 1\right) - 0 = 2 - \frac{\pi}{2} \end{aligned}$$



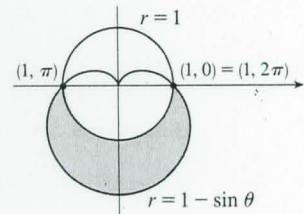
23.  $2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$ .

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2}[(2 \cos \theta)^2 - 1^2] d\theta = \int_0^{\pi/3} (4 \cos^2 \theta - 1) d\theta \\ &= \int_0^{\pi/3} \{4 \left[\frac{1}{2}(1 + \cos 2\theta)\right] - 1\} d\theta = \int_0^{\pi/3} (1 + 2 \cos 2\theta) d\theta \\ &= [\theta + \sin 2\theta]_0^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$



24.  $1 - \sin \theta = 1 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow$

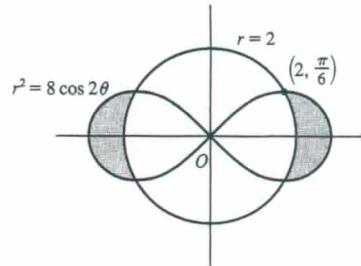
$$\begin{aligned} A &= \int_{\pi}^{2\pi} \frac{1}{2}[(1 - \sin \theta)^2 - 1] d\theta = \frac{1}{2} \int_{\pi}^{2\pi} (\sin^2 \theta - 2 \sin \theta) d\theta \\ &= \frac{1}{4} \int_{\pi}^{2\pi} (1 - \cos 2\theta - 4 \sin \theta) d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta + 4 \cos \theta\right]_{\pi}^{2\pi} \\ &= \frac{1}{4}\pi + 2 \end{aligned}$$



25. To find the area inside the lemniscate  $r^2 = 8 \cos 2\theta$  and outside the circle  $r = 2$ ,

we first note that the two curves intersect when  $r^2 = 8 \cos 2\theta$  and  $r = 2$ ,  
that is, when  $\cos 2\theta = \frac{1}{2}$ . For  $-\pi < \theta \leq \pi$ ,  $\cos 2\theta = \frac{1}{2} \Leftrightarrow 2\theta = \pm\pi/3$   
or  $\pm\pi/3 \Leftrightarrow \theta = \pm\pi/6$  or  $\pm 5\pi/6$ . The figure shows that the desired area is  
4 times the area between the curves from 0 to  $\pi/6$ . Thus,

$$\begin{aligned} A &= 4 \int_0^{\pi/6} \left[ \frac{1}{2}(8 \cos 2\theta) - \frac{1}{2}(2)^2 \right] d\theta = 8 \int_0^{\pi/6} (2 \cos 2\theta - 1) d\theta \\ &= 8 \left[ \sin 2\theta - \theta \right]_0^{\pi/6} = 8(\sqrt{3}/2 - \pi/6) = 4\sqrt{3} - 4\pi/3 \end{aligned}$$

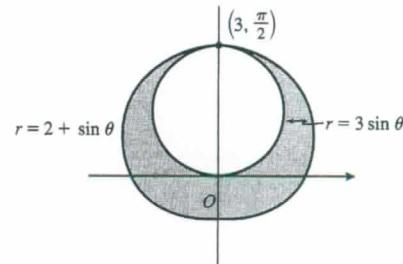


26. To find the shaded area  $A$ , we'll find the area  $A_1$  inside the curve  $r = 2 + \sin \theta$

and subtract  $\pi(\frac{3}{2})^2$  since  $r = 3 \sin \theta$  is a circle with radius  $\frac{3}{2}$ .

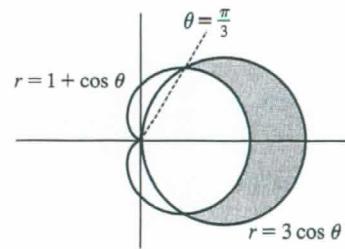
$$\begin{aligned} A_1 &= \int_0^{2\pi} \frac{1}{2}(2 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta + \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [4 + 4 \sin \theta + \frac{1}{2} \cdot (1 - \cos 2\theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (\frac{9}{2} + 4 \sin \theta - \frac{1}{2} \cos 2\theta) d\theta \\ &= \frac{1}{2} [\frac{9}{2}\theta - 4 \cos \theta - \frac{1}{4} \sin 2\theta]_0^{2\pi} = \frac{1}{2} [(9\pi - 4) - (-4)] = \frac{9\pi}{2} \end{aligned}$$

$$\text{So } A = A_1 - \frac{9\pi}{2} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}.$$



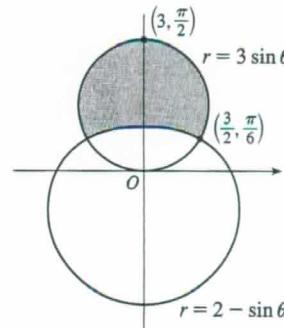
27.  $3 \cos \theta = 1 + \cos \theta \Leftrightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$ .

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2}[(3 \cos \theta)^2 - (1 + \cos \theta)^2] d\theta \\ &= \int_0^{\pi/3} (8 \cos^2 \theta - 2 \cos \theta - 1) d\theta = \int_0^{\pi/3} [4(1 + \cos 2\theta) - 2 \cos \theta - 1] d\theta \\ &= \int_0^{\pi/3} (3 + 4 \cos 2\theta - 2 \cos \theta) d\theta = [3\theta + 2 \sin 2\theta - 2 \sin \theta]_0^{\pi/3} \\ &= \pi + \sqrt{3} - \sqrt{3} = \pi \end{aligned}$$



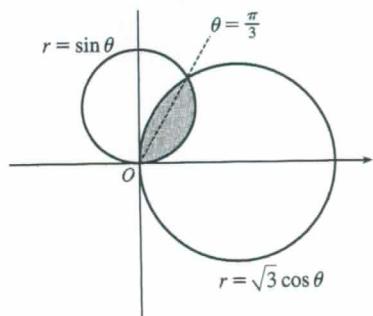
28.  $3 \sin \theta = 2 - \sin \theta \Rightarrow 4 \sin \theta = 2 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$ .

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} \frac{1}{2}[(3 \sin \theta)^2 - (2 - \sin \theta)^2] d\theta \\ &= \int_{\pi/6}^{\pi/2} (9 \sin^2 \theta - 4 + 4 \sin \theta - \sin^2 \theta) d\theta \\ &= \int_{\pi/6}^{\pi/2} (8 \sin^2 \theta + 4 \sin \theta - 4) d\theta \\ &= 4 \int_{\pi/6}^{\pi/2} [2 \cdot \frac{1}{2}(1 - \cos 2\theta) + \sin \theta - 1] d\theta \\ &= 4 \int_{\pi/6}^{\pi/2} (\sin \theta - \cos 2\theta) d\theta = 4[-\cos \theta - \frac{1}{2} \sin 2\theta]_{\pi/6}^{\pi/2} \\ &= 4[(0 - 0) - \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)] = 4\left(\frac{3\sqrt{3}}{4}\right) = 3\sqrt{3} \end{aligned}$$

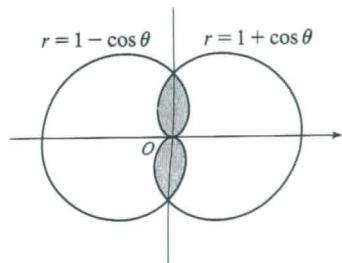


29.  $\sqrt{3} \cos \theta = \sin \theta \Rightarrow \sqrt{3} = \frac{\sin \theta}{\cos \theta} \Rightarrow \tan \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ .

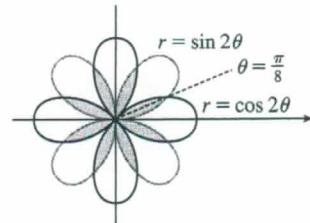
$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} (\sin \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} (\sqrt{3} \cos \theta)^2 d\theta \\ &= \int_0^{\pi/3} \frac{1}{2} \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} \cdot 3 \cdot \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/3} + \frac{3}{4} [\theta + \frac{1}{2} \sin 2\theta]_{\pi/3}^{\pi/2} \\ &= \frac{1}{4} \left[ \left( \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) - 0 \right] + \frac{3}{4} \left[ \left( \frac{\pi}{2} + 0 \right) - \left( \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) \right] \\ &= \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{\pi}{8} - \frac{3\sqrt{3}}{16} = \frac{5\pi}{24} - \frac{\sqrt{3}}{4} \end{aligned}$$



30.  $A = 4 \int_0^{\pi/2} \frac{1}{2} (1 - \cos \theta)^2 d\theta = 2 \int_0^{\pi/2} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$   
 $= 2 \int_0^{\pi/2} [1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta$   
 $= 2 \int_0^{\pi/2} (\frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta) d\theta = \int_0^{\pi/2} (3 - 4 \cos \theta + \cos 2\theta) d\theta$   
 $= [3\theta - 4 \sin \theta + \frac{1}{2} \sin 2\theta]_0^{\pi/2} = \frac{3\pi}{2} - 4$

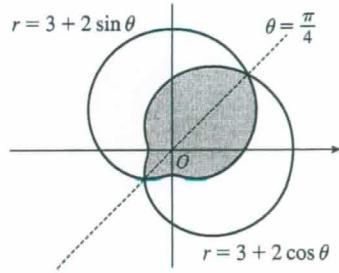


31.  $\sin 2\theta = \cos 2\theta \Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{8} \Rightarrow$   
 $A = 8 \cdot 2 \int_0^{\pi/8} \frac{1}{2} \sin^2 2\theta d\theta = 8 \int_0^{\pi/8} \frac{1}{2} (1 - \cos 4\theta) d\theta$   
 $= 4[\theta - \frac{1}{4} \sin 4\theta]_0^{\pi/8} = 4(\frac{\pi}{8} - \frac{1}{4} \cdot 1) = \frac{\pi}{2} - 1$



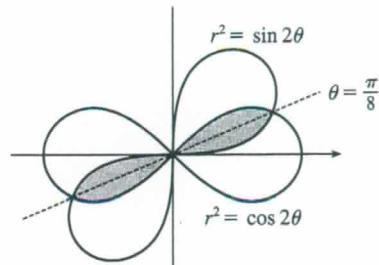
32.  $3 + 2 \cos \theta = 3 + 2 \sin \theta \Rightarrow \cos \theta = \sin \theta \Rightarrow \theta = \frac{\pi}{4}$  or  $\frac{5\pi}{4}$ .

$$\begin{aligned} A &= 2 \int_{\pi/4}^{5\pi/4} \frac{1}{2} (3 + 2 \cos \theta)^2 d\theta = \int_{\pi/4}^{5\pi/4} (9 + 12 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \int_{\pi/4}^{5\pi/4} [9 + 12 \cos \theta + 4 \cdot \frac{1}{2}(1 + \cos 2\theta)] d\theta \\ &= \int_{\pi/4}^{5\pi/4} (11 + 12 \cos \theta + 2 \cos 2\theta) d\theta = [11\theta + 12 \sin \theta + \sin 2\theta]_{\pi/4}^{5\pi/4} \\ &= (\frac{55\pi}{4} - 6\sqrt{2} + 1) - (\frac{11\pi}{4} + 6\sqrt{2} + 1) = 11\pi - 12\sqrt{2} \end{aligned}$$



33.  $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{8}$

$$\begin{aligned} A &= 4 \int_0^{\pi/8} \frac{1}{2} \sin 2\theta d\theta \quad [\text{since } r^2 = \sin 2\theta] \\ &= \int_0^{\pi/8} 2 \sin 2\theta d\theta = [-\cos 2\theta]_0^{\pi/8} \\ &= -\frac{1}{2}\sqrt{2} - (-1) = 1 - \frac{1}{2}\sqrt{2} \end{aligned}$$



34. Let  $\alpha = \tan^{-1}(b/a)$ . Then

$$\begin{aligned} A &= \int_0^\alpha \frac{1}{2}(a \sin \theta)^2 d\theta + \int_\alpha^{\pi/2} \frac{1}{2}(b \cos \theta)^2 d\theta \\ &= \frac{1}{4}a^2 \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^\alpha + \frac{1}{4}b^2 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_\alpha^{\pi/2} \\ &= \frac{1}{4}\alpha(a^2 - b^2) + \frac{1}{8}\pi b^2 - \frac{1}{4}(a^2 + b^2)(\sin \alpha \cos \alpha) \\ &= \frac{1}{4}(a^2 - b^2) \tan^{-1}(b/a) + \frac{1}{8}\pi b^2 - \frac{1}{4}ab \end{aligned}$$

35. The darker shaded region (from  $\theta = 0$  to  $\theta = 2\pi/3$ ) represents  $\frac{1}{2}$  of the desired area plus  $\frac{1}{2}$  of the area of the inner loop.

From this area, we'll subtract  $\frac{1}{2}$  of the area of the inner loop (the lighter shaded region from  $\theta = 2\pi/3$  to  $\theta = \pi$ ), and then double that difference to obtain the desired area.

$$\begin{aligned} A &= 2 \left[ \int_0^{2\pi/3} \frac{1}{2} \left( \frac{1}{2} + \cos \theta \right)^2 d\theta - \int_{2\pi/3}^\pi \frac{1}{2} \left( \frac{1}{2} + \cos \theta \right)^2 d\theta \right] \\ &= \int_0^{2\pi/3} \left( \frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta - \int_{2\pi/3}^\pi \left( \frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta \\ &= \int_0^{2\pi/3} \left[ \frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &\quad - \int_{2\pi/3}^\pi \left[ \frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= \left[ \frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi/3} - \left[ \frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{2\pi/3}^\pi \\ &= \left( \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) - \left( \frac{\pi}{4} + \frac{\pi}{2} \right) + \left( \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) \\ &= \frac{\pi}{4} + \frac{3}{4}\sqrt{3} = \frac{1}{4}(\pi + 3\sqrt{3}) \end{aligned}$$

36.  $r = 0 \Rightarrow 1 + 2 \cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$  [for  $0 \leq 3\theta \leq 2\pi$ ]  $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$ . The darker shaded region (from  $\theta = 0$  to  $\theta = 2\pi/9$ ) represents  $\frac{1}{2}$  of the desired area plus  $\frac{1}{2}$  of the area of the inner loop. From this area, we'll subtract  $\frac{1}{2}$  of the area of the inner loop (the lighter shaded region from  $\theta = 2\pi/9$  to  $\theta = \pi/3$ ), and then double that difference to obtain the desired area.

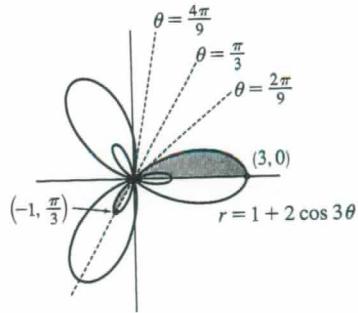
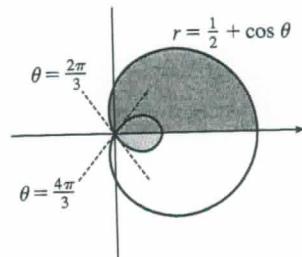
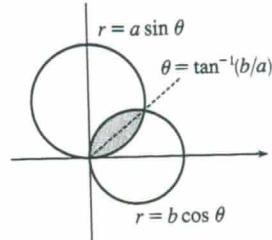
$$A = 2 \left[ \int_0^{2\pi/9} \frac{1}{2}(1 + 2 \cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2}(1 + 2 \cos 3\theta)^2 d\theta \right]$$

Now

$$\begin{aligned} r^2 &= (1 + 2 \cos 3\theta)^2 = 1 + 4 \cos 3\theta + 4 \cos^2 3\theta = 1 + 4 \cos 3\theta + 4 \cdot \frac{1}{2}(1 + \cos 6\theta) \\ &= 1 + 4 \cos 3\theta + 2 + 2 \cos 6\theta = 3 + 4 \cos 3\theta + 2 \cos 6\theta \end{aligned}$$

and  $\int r^2 d\theta = 3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta + C$ , so

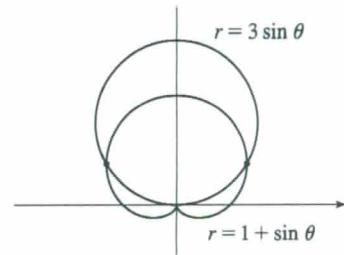
$$\begin{aligned} A &= \left[ 3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta \right]_0^{2\pi/9} - \left[ 3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta \right]_{2\pi/9}^{\pi/3} \\ &= \left[ \left( \frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) - 0 \right] - \left[ (\pi + 0 + 0) - \left( \frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) \right] \\ &= \frac{4\pi}{3} + \frac{4}{3}\sqrt{3} - \frac{1}{3}\sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3} \end{aligned}$$



37. The pole is a point of intersection.

$$1 + \sin \theta = 3 \sin \theta \Rightarrow 1 = 2 \sin \theta \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}.$$

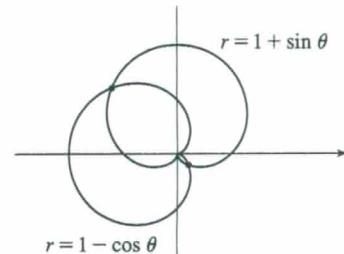
The other two points of intersection are  $(\frac{3}{2}, \frac{\pi}{6})$  and  $(\frac{3}{2}, \frac{5\pi}{6})$ .



38. The pole is a point of intersection.

$$1 - \cos \theta = 1 + \sin \theta \Rightarrow -\cos \theta = \sin \theta \Rightarrow -1 = \tan \theta \Rightarrow \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

The other two points of intersection are  $(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4})$  and  $(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4})$ .



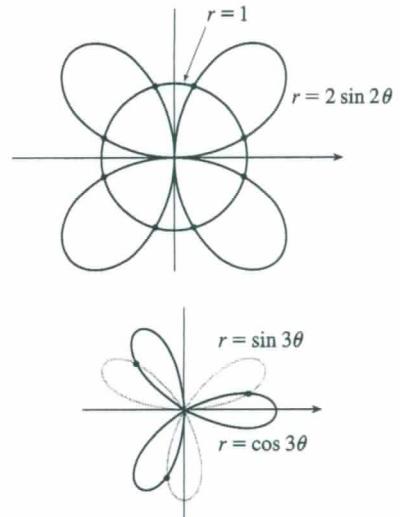
39.  $2 \sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{17\pi}{6}$ .

By symmetry, the eight points of intersection are given by

$(1, \theta)$ , where  $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \text{ and } \frac{17\pi}{12}$ , and

$(-1, \theta)$ , where  $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \text{ and } \frac{23\pi}{12}$ .

[There are many ways to describe these points.]

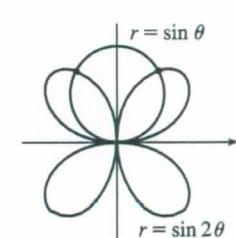


40. Clearly the pole lies on both curves.  $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow$

$$3\theta = \frac{\pi}{4} + n\pi \quad [n \text{ any integer}] \Rightarrow \theta = \frac{\pi}{12} + \frac{\pi}{3}n \Rightarrow$$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \text{ or } \frac{3\pi}{4}$ , so the three remaining intersection points are  $(\frac{1}{\sqrt{2}}, \frac{\pi}{12})$ ,

$(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12})$ , and  $(\frac{1}{\sqrt{2}}, \frac{3\pi}{4})$ .

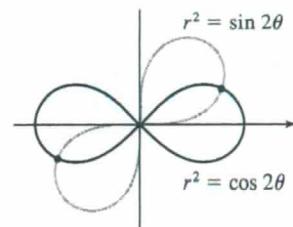


41. The pole is a point of intersection.  $\sin \theta = \sin 2\theta = 2 \sin \theta \cos \theta \Leftrightarrow$

$$\sin \theta (1 - 2 \cos \theta) = 0 \Leftrightarrow \sin \theta = 0 \text{ or } \cos \theta = \frac{1}{2} \Rightarrow$$

$\theta = 0, \pi, \frac{\pi}{3}, \text{ or } -\frac{\pi}{3} \Rightarrow$  the other intersection points are  $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$

and  $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$  [by symmetry].

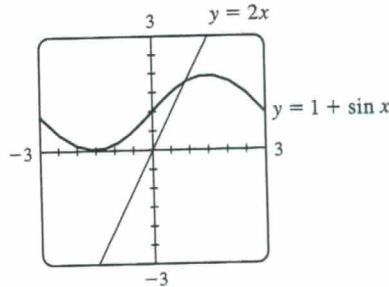
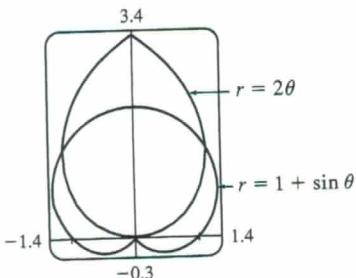


42. Clearly the pole is a point of intersection.  $\sin 2\theta = \cos 2\theta \Rightarrow$

$$\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi \quad [\text{since } \sin 2\theta \text{ and } \cos 2\theta \text{ must be positive in the equations}] \Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8} \text{ or } \frac{9\pi}{8}.$$

So the curves also intersect at  $(\frac{1}{\sqrt{2}}, \frac{\pi}{8})$  and  $(\frac{1}{\sqrt{2}}, \frac{9\pi}{8})$ .

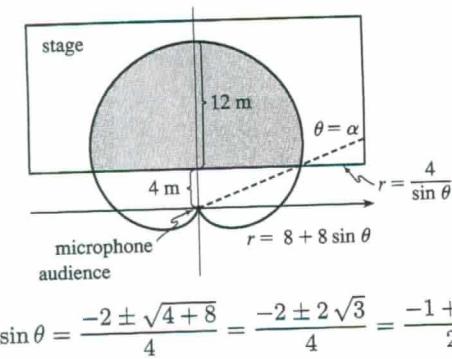
43.



From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the  $\theta$ -values of the intersection points to be  $\alpha \approx 0.88786 \approx 0.89$  and  $\pi - \alpha \approx 2.25$ . (The first of these values may be more easily estimated by plotting  $y = 1 + \sin x$  and  $y = 2x$  in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$\begin{aligned} A &= 2 \int_0^\alpha \frac{1}{2}(2\theta)^2 d\theta + 2 \int_\alpha^{\pi/2} \frac{1}{2}(1 + \sin \theta)^2 d\theta = \int_0^\alpha 4\theta^2 d\theta + \int_\alpha^{\pi/2} [1 + 2\sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta \\ &= [\frac{4}{3}\theta^3]_0^\alpha + [\theta - 2\cos \theta + (\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)]_\alpha^{\pi/2} = \frac{4}{3}\alpha^3 + [(\frac{\pi}{2} + \frac{\pi}{4}) - (\alpha - 2\cos \alpha + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha)] \approx 3.4645 \end{aligned}$$

44.



We need to find the shaded area  $A$  in the figure. The horizontal line representing the front of the stage has equation  $y = 4 \Leftrightarrow r \sin \theta = 4 \Rightarrow r = 4/\sin \theta$ . This line intersects the curve  $r = 8 + 8 \sin \theta$  when  $8 + 8 \sin \theta = \frac{4}{\sin \theta} \Rightarrow 8 \sin \theta + 8 \sin^2 \theta = 4 \Rightarrow 2 \sin^2 \theta + 2 \sin \theta - 1 = 0 \Rightarrow$

$$\sin \theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2} \quad [\text{the other value is less than } -1] \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

This angle is about  $21.5^\circ$  and is denoted by  $\alpha$  in the figure.

$$\begin{aligned} A &= 2 \int_\alpha^{\pi/2} \frac{1}{2}(8 + 8 \sin \theta)^2 d\theta - 2 \int_\alpha^{\pi/2} \frac{1}{2}(4 \csc \theta)^2 d\theta = 64 \int_\alpha^{\pi/2} (1 + 2 \sin \theta + \sin^2 \theta) d\theta - 16 \int_\alpha^{\pi/2} \csc^2 \theta d\theta \\ &= 64 \int_\alpha^{\pi/2} (1 + 2 \sin \theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta) d\theta + 16 \int_\alpha^{\pi/2} (-\csc^2 \theta) d\theta = 64[\frac{3}{2}\theta - 2\cos \theta - \frac{1}{4}\sin 2\theta]_\alpha^{\pi/2} + 16[\cot \theta]_\alpha^{\pi/2} \\ &= 16[6\theta - 8\cos \theta - \sin 2\theta + \cot \theta]_\alpha^{\pi/2} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos \alpha - \sin 2\alpha + \cot \alpha)] \\ &= 48\pi - 96\alpha + 128\cos \alpha + 16\sin 2\alpha - 16\cot \alpha \end{aligned}$$

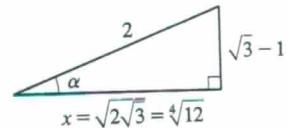
From the figure,  $x^2 + (\sqrt{3}-1)^2 = 2^2 \Rightarrow x^2 = 4 - (3 - 2\sqrt{3} + 1) \Rightarrow$

$x^2 = 2\sqrt{3} = \sqrt{12}$ , so  $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$ . Using the trigonometric relationships

for a right triangle and the identity  $\sin 2\alpha = 2\sin \alpha \cos \alpha$ , we continue:

$$\begin{aligned} A &= 48\pi - 96\alpha + 128 \cdot \frac{\sqrt[4]{12}}{2} + 16 \cdot 2 \cdot \frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt[4]{12}}{2} - 16 \cdot \frac{\sqrt[4]{12}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} \\ &= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}(\sqrt{3}-1) - 8\sqrt[4]{12}(\sqrt{3}+1) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \end{aligned}$$

$$\approx 204.16 \text{ m}^2$$



$$\begin{aligned}
 45. L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi/3} \sqrt{(3 \sin \theta)^2 + (3 \cos \theta)^2} d\theta = \int_0^{\pi/3} \sqrt{9(\sin^2 \theta + \cos^2 \theta)} d\theta \\
 &= 3 \int_0^{\pi/3} d\theta = 3[\theta]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) = \pi.
 \end{aligned}$$

As a check, note that the circumference of a circle with radius  $\frac{3}{2}$  is  $2\pi\left(\frac{3}{2}\right) = 3\pi$ , and since  $\theta = 0$  to  $\pi = \frac{\pi}{3}$  traces out  $\frac{1}{3}$  of the circle (from  $\theta = 0$  to  $\theta = \pi$ ),  $\frac{1}{3}(3\pi) = \pi$ .

$$\begin{aligned}
 46. L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2} d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta = \int_0^{2\pi} \sqrt{5e^{4\theta}} d\theta \\
 &= \sqrt{5} \int_0^{2\pi} e^{2\theta} d\theta = \frac{\sqrt{5}}{2} [e^{2\theta}]_0^{2\pi} = \frac{\sqrt{5}}{2}(e^{4\pi} - 1)
 \end{aligned}$$

$$\begin{aligned}
 47. L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta \\
 &= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)} d\theta = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta
 \end{aligned}$$

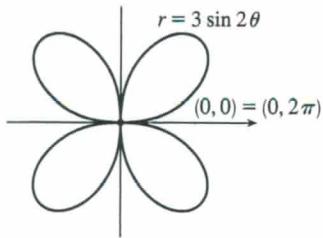
Now let  $u = \theta^2 + 4$ , so that  $du = 2\theta d\theta$  [ $\theta d\theta = \frac{1}{2} du$ ] and

$$\int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_4^{4\pi^2+4} \frac{1}{2} \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} [u^{3/2}]_4^{4(\pi^2+1)} = \frac{1}{3}[4^{3/2}(\pi^2 + 1)^{3/2} - 4^{3/2}] = \frac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$$

$$\begin{aligned}
 48. L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta \stackrel{?}{=} \left[ \frac{\theta}{2} \sqrt{\theta^2 + 1} + \frac{1}{2} \ln(\theta + \sqrt{\theta^2 + 1}) \right]_0^{2\pi} \\
 &= \pi\sqrt{4\pi^2 + 1} + \frac{1}{2} \ln(2\pi + \sqrt{4\pi^2 + 1})
 \end{aligned}$$

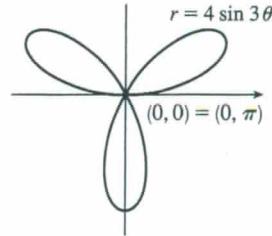
49. The curve  $r = 3 \sin 2\theta$  is completely traced with  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned}
 r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (3 \sin 2\theta)^2 + (6 \cos 2\theta)^2 \Rightarrow \\
 L &= \int_0^{2\pi} \sqrt{9 \sin^2 2\theta + 36 \cos^2 2\theta} d\theta \approx 29.0653
 \end{aligned}$$



50. The curve  $r = 4 \sin 3\theta$  is completely traced with  $0 \leq \theta \leq \pi$ .

$$\begin{aligned}
 r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (4 \sin 3\theta)^2 + (12 \cos 3\theta)^2 \Rightarrow \\
 L &= \int_0^\pi \sqrt{16 \sin^2 3\theta + 144 \cos^2 3\theta} d\theta \approx 26.7298
 \end{aligned}$$

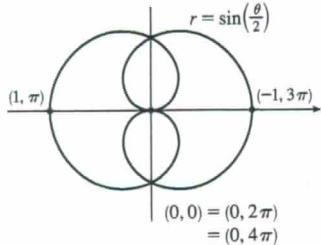


51. The curve  $r = \sin\left(\frac{\theta}{2}\right)$  is completely traced with  $0 \leq \theta \leq 4\pi$ .

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2\left(\frac{\theta}{2}\right) + \left[\frac{1}{2} \cos\left(\frac{\theta}{2}\right)\right]^2 \Rightarrow$$

$$L = \int_0^{4\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right) + \frac{1}{4} \cos^2\left(\frac{\theta}{2}\right)} d\theta$$

$$\approx 9.6884$$

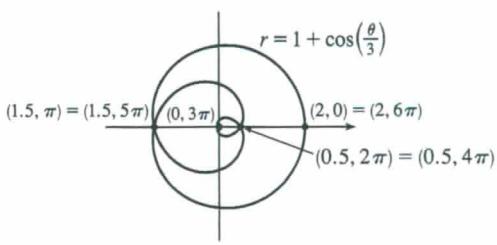


52. The curve  $r = 1 + \cos\left(\frac{\theta}{3}\right)$  is completely traced with  $0 \leq \theta \leq 6\pi$ .

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = [1 + \cos\left(\frac{\theta}{3}\right)]^2 + [-\frac{1}{3} \sin\left(\frac{\theta}{3}\right)]^2 \Rightarrow$$

$$L = \int_0^{6\pi} \sqrt{[1 + \cos\left(\frac{\theta}{3}\right)]^2 + \frac{1}{9} \sin^2\left(\frac{\theta}{3}\right)} d\theta$$

$$\approx 19.6676$$



53. The curve  $r = \cos^4(\theta/4)$  is completely traced with  $0 \leq \theta \leq 4\pi$ .

$$r^2 + (dr/d\theta)^2 = [\cos^4(\theta/4)]^2 + [4 \cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \frac{1}{4}]^2$$

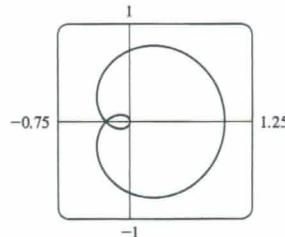
$$= \cos^8(\theta/4) + \cos^6(\theta/4) \sin^2(\theta/4)$$

$$= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)$$

$$L = \int_0^{4\pi} \sqrt{\cos^6(\theta/4)} d\theta = \int_0^{4\pi} |\cos^3(\theta/4)| d\theta$$

$$= 2 \int_0^{2\pi} \cos^3(\theta/4) d\theta \quad [\text{since } \cos^3(\theta/4) \geq 0 \text{ for } 0 \leq \theta \leq 2\pi] \quad = 8 \int_0^{\pi/2} \cos^3 u du \quad [u = \frac{1}{4}\theta]$$

$$\stackrel{68}{=} 8 \left[ \frac{1}{3}(2 + \cos^2 u) \sin u \right]_0^{\pi/2} = \frac{8}{3}[(2 \cdot 1) - (3 \cdot 0)] = \frac{16}{3}$$



54. The curve  $r = \cos^2(\theta/2)$  is completely traced with  $0 \leq \theta \leq 2\pi$ .

$$r^2 + (dr/d\theta)^2 = [\cos^2(\theta/2)]^2 + [2 \cos(\theta/2) \cdot (-\sin(\theta/2)) \cdot \frac{1}{2}]^2$$

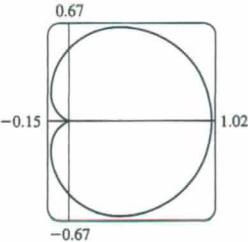
$$= \cos^4(\theta/2) + \cos^2(\theta/2) \sin^2(\theta/2)$$

$$= \cos^2(\theta/2)[\cos^2(\theta/2) + \sin^2(\theta/2)]$$

$$= \cos^2(\theta/2)$$

$$L = \int_0^{2\pi} \sqrt{\cos^2(\theta/2)} d\theta = \int_0^{2\pi} |\cos(\theta/2)| d\theta = 2 \int_0^\pi \cos(\theta/2) d\theta \quad [\text{since } \cos(\theta/2) \geq 0 \text{ for } 0 \leq \theta \leq \pi]$$

$$= 4 \int_0^{\pi/2} \cos u du \quad [u = \frac{1}{2}\theta] \quad = 4[\sin u]_0^{\pi/2} = 4(1 - 0) = 4$$



55. (a) From (10.2.7),

$$S = \int_a^b 2\pi y \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta$$

$$= \int_a^b 2\pi y \sqrt{r^2 + (dr/d\theta)^2} d\theta \quad [\text{from the derivation of Equation 10.4.5}]$$

$$= \int_a^b 2\pi r \sin \theta \sqrt{r^2 + (dr/d\theta)^2} d\theta$$

- (b) The curve  $r^2 = \cos 2\theta$  goes through the pole when  $\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ . We'll rotate the curve from  $\theta = 0$  to  $\theta = \frac{\pi}{4}$  and double this value to obtain the total surface area generated.

$$r^2 = \cos 2\theta \Rightarrow 2r \frac{dr}{d\theta} = -2 \sin 2\theta \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2} = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

$$S = 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \sin \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} d\theta = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} d\theta \\ = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \frac{1}{\sqrt{\cos 2\theta}} d\theta = 4\pi \int_0^{\pi/4} \sin \theta d\theta = 4\pi [-\cos \theta]_0^{\pi/4} = -4\pi \left(\frac{\sqrt{2}}{2} - 1\right) = 2\pi(2 - \sqrt{2})$$

56. (a) Rotation around  $\theta = \frac{\pi}{2}$  is the same as rotation around the  $y$ -axis, that is,  $S = \int_a^b 2\pi x ds$  where

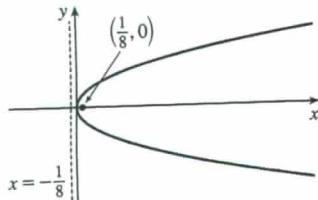
$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$  for a parametric equation, and for the special case of a polar equation,  $x = r \cos \theta$  and  $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = \sqrt{r^2 + (dr/d\theta)^2} d\theta$  [see the derivation of Equation 10.4.5]. Therefore, for a polar equation rotated around  $\theta = \frac{\pi}{2}$ ,  $S = \int_a^b 2\pi r \cos \theta \sqrt{r^2 + (dr/d\theta)^2} d\theta$ .

- (b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from  $\theta = 0$  to  $\theta = \frac{\pi}{4}$  to obtain the total surface area.

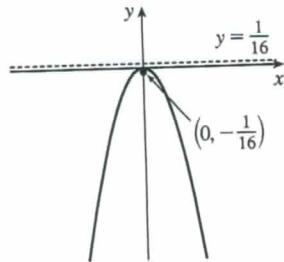
$$S = 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \cos \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} d\theta = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} d\theta \\ = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \frac{1}{\sqrt{\cos 2\theta}} d\theta = 4\pi \int_0^{\pi/4} \cos \theta d\theta = 4\pi [\sin \theta]_0^{\pi/4} = 4\pi \left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\pi$$

## 10.5 Conic Sections

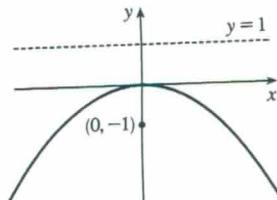
1.  $x = 2y^2 \Rightarrow y^2 = \frac{1}{2}x$ .  $4p = \frac{1}{2}$ , so  $p = \frac{1}{8}$ . The vertex is  $(0, 0)$ , the focus is  $(\frac{1}{8}, 0)$ , and the directrix is  $x = -\frac{1}{8}$ .



3.  $4x^2 = -y \Rightarrow x^2 = -\frac{1}{4}y$ .  $4p = -\frac{1}{4}$ , so  $p = -\frac{1}{16}$ . The vertex is  $(0, 0)$ , the focus is  $(0, -\frac{1}{16})$ , and the directrix is  $y = \frac{1}{16}$ .



2.  $4y + x^2 = 0 \Rightarrow x^2 = -4y$ .  $4p = -4$ , so  $p = -1$ . The vertex is  $(0, 0)$ , the focus is  $(0, -1)$ , and the directrix is  $y = 1$ .



4.  $y^2 = 12x$ .  $4p = 12$ , so  $p = 3$ . The vertex is  $(0, 0)$ , the focus is  $(3, 0)$ , and the directrix is  $x = -3$ .

