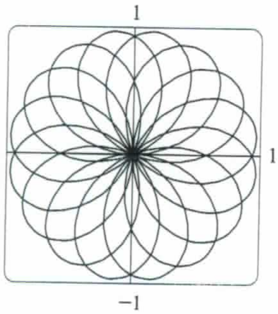


FIGURE 18
$r = \sin(8\theta/5)$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops. □

EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

■ In Exercise 55 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

SOLUTION Figure 19 shows computer-drawn graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

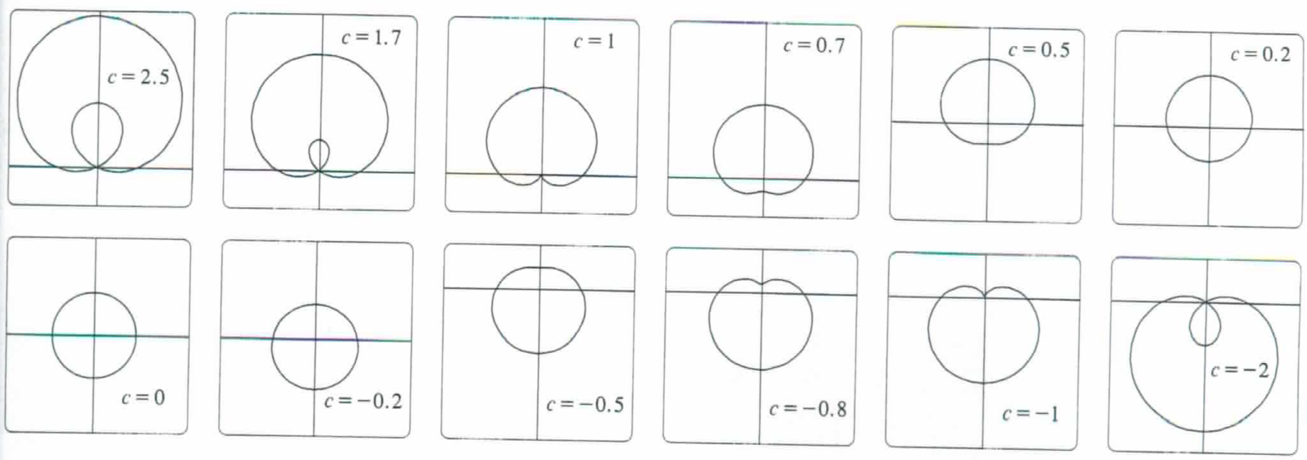


FIGURE 19
Members of the family of limaçons $r = 1 + c\sin\theta$

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. □

10.3 EXERCISES

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(2, \pi/3)$ (b) $(1, -3\pi/4)$ (c) $(-1, \pi/2)$

2. (a) $(1, 7\pi/4)$ (b) $(-3, \pi/6)$ (c) $(1, -1)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(1, \pi)$ (b) $(2, -2\pi/3)$ (c) $(-2, 3\pi/4)$

4. (a) $(-\sqrt{2}, 5\pi/4)$ (b) $(1, 5\pi/2)$ (c) $(2, -7\pi/6)$

5–6 The Cartesian coordinates of a point are given.
(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.
(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(2, -2)$ (b) $(-1, \sqrt{3})$

6. (a) $(3\sqrt{3}, 3)$ (b) $(1, -2)$

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 \leqslant r \leqslant 2$

8. $r \geqslant 0, \quad \pi/3 \leqslant \theta \leqslant 2\pi/3$

9. $0 \leqslant r < 4, \quad -\pi/2 \leqslant \theta < \pi/6$

10. $2 < r \leqslant 5, \quad 3\pi/4 < \theta < 5\pi/4$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $r \geqslant 1, \quad \pi \leqslant \theta \leqslant 2\pi$

13. Find the distance between the points with polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$.

14. Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

15–20 Identify the curve by finding a Cartesian equation for the curve.

15. $r = 2$

16. $r \cos\theta = 1$

17. $r = 3\sin\theta$

18. $r = 2\sin\theta + 2\cos\theta$

19. $r = \csc\theta$

20. $r = \tan\theta \sec\theta$

21–26 Find a polar equation for the curve represented by the given Cartesian equation.

21. $x = 3$

22. $x^2 + y^2 = 9$

23. $x = -y^2$

24. $x + y = 9$

25. $x^2 + y^2 = 2cx$

26. $xy = 4$

27–28 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

27. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point (3, 3)

28. (a) A circle with radius 5 and center (2, 3)
(b) A circle centered at the origin with radius 4

29–48 Sketch the curve with the given polar equation.

29. $\theta = -\pi/6$

30. $r^2 - 3r + 2 = 0$

31. $r = \sin\theta$

32. $r = -3\cos\theta$

33. $r = 2(1 - \sin\theta), \ \theta \geqslant 0$

34. $r = 1 - 3\cos\theta$

35. $r = \theta, \ \theta \geqslant 0$

36. $r = \ln\theta, \ \theta \geqslant 1$

37. $r = 4\sin 3\theta$

38. $r = \cos 5\theta$

39. $r = 2\cos 4\theta$

40. $r = 3\cos 6\theta$

41. $r = 1 - 2\sin\theta$

42. $r = 2 + \sin\theta$

43. $r^2 = 9\sin 2\theta$

44. $r^2 = \cos 4\theta$

45. $r = 2\cos(3\theta/2)$

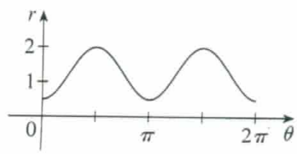
46. $r^2\theta = 1$

47. $r = 1 + 2\cos 2\theta$

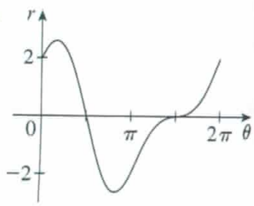
48. $r = 1 + 2\cos(\theta/2)$

49–50 The figure shows the graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

49.



50.



51. Show that the polar curve $r = 4 + 2\sec\theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \to \pm\infty} x = 2$. Use this fact to help sketch the conchoid.

52. Show that the curve $r = 2 - \csc\theta$ (also a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r \to \pm\infty} y = -1$. Use this fact to help sketch the conchoid.

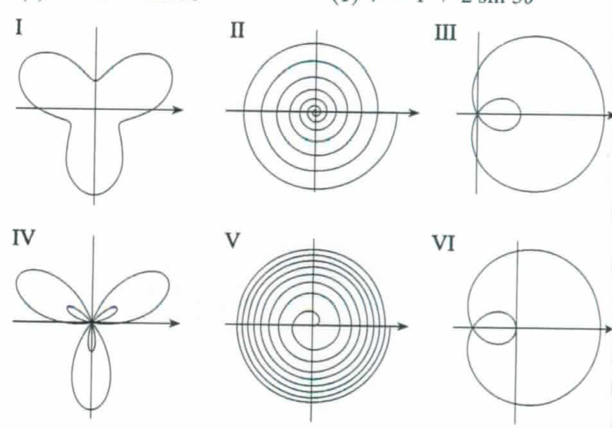
53. Show that the curve $r = \sin\theta \tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

54. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

55. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 19 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

56. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sqrt{\theta}, \quad 0 \leqslant \theta \leqslant 16\pi$
(b) $r = \theta^2, \quad 0 \leqslant \theta \leqslant 16\pi$
(c) $r = \cos(\theta/3)$
(d) $r = 1 + 2\cos\theta$
(e) $r = 2 + \sin 3\theta$
(f) $r = 1 + 2\sin 3\theta$



57–62 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

57. $r = 2\sin\theta, \quad \theta = \pi/6$

58. $r = 2 - \sin\theta, \quad \theta = \pi/3$

59. $r = 1/\theta, \quad \theta = \pi$

60. $r = \cos(\theta/3), \quad \theta = \pi$

61. $r = \cos 2\theta, \quad \theta = \pi/4$

62. $r = 1 + 2\cos\theta, \quad \theta = \pi/3$

63–68 Find the points on the given curve where the tangent line is horizontal or vertical.

63. $r = 3\cos\theta$

64. $r = 1 - \sin\theta$

65. $r = 1 + \cos\theta$

66. $r = e^{\theta}$

67. $r = 2 + \sin\theta$

68. $r^2 = \sin 2\theta$

69. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

70. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

71–76 Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

71. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

72. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

73. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

74. $r = \sin^2(4\theta) + \cos(4\theta)$

75. $r = 2 - 5\sin(\theta/6)$

76. $r = \cos(\theta/2) + \cos(\theta/3)$

77. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

78. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

79. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?
(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

80. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

81. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

82. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

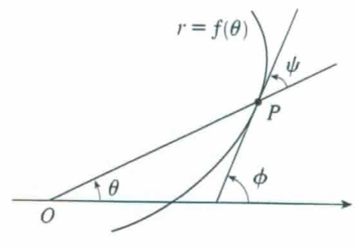
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

83. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



84. (a) Use Exercise 83 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.
(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.
(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.