Def. A sequence of real numbers is an ordered list of real numbers. Common notations for sequences are:

$$\{a_n\}_{n=1}^{\infty} \overset{\text{or}}{=} \{a_1, a_2, a_3, a_4, \ldots\} \qquad \text{or} \qquad \{a_n\}_{n=17}^{\infty} \overset{\text{or}}{=} \{a_{17}, a_{18}, a_{19}, a_{20}, \ldots\}.$$

We can also think of a sequence $\{a_n\}_{n=1}^{\infty}$ as a function $f\colon \mathbb{N} \to \mathbb{R}$ where $\boxed{f(n) = a_n}$.

Ex. 1. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \overset{\text{or}}{=} \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Here $a_n = \frac{1}{n}$.
We can also view the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ as the function $f\colon \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{1}{n}$ for $n \in \mathbb{N}$.
We can extend the domain of f to get the function $f\colon [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ for $x \in [1, \infty)$.
Let's draw a picture.

Guess: $\lim_{n\to\infty} \frac{1}{n} = \boxed{}$.

Limit of a Function -vs- Limit of a Sequence

Now consider a function $f\colon [1, \infty) \to \mathbb{R}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ related by $f(n) = a_n$.
So we can view this sequence $\{a_n\}_{n=1}^{\infty}$ as the function $f\colon \mathbb{N} \to \mathbb{R}$.
In Calc. I, we learned how to take a limit of a function.

Calc.I. The limit of the function $y = f(x)$ as $x \to \infty$ is L, which is written $\lim_{x\to\infty} f(x) = L$ or $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = L$

In Calc. II, we now learn how to take a limit of a sequence.

Calc.II. The limit of the sequence $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ is L, which is written $\lim_{n\to\infty} a_n = L$ or $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} a_n = L$.

Theses two limits concepts are closely related. So first let's remind ourselves the definiton of limit of a function and then do the necessary modifications to get the definition of limit of a sequence. We will have 3 cases:

- Case 1. $L \in \mathbb{R}$, i.e., L is some (finite) real number
- Case 2. $L = \infty$
- Case 3. $L = -\infty$.

Defintions of: Limit of a Function & Limit of a Sequence

Case $L \in \mathbb{R}$.

1fun. $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = L \in \mathbb{R} \Leftrightarrow$ for each $\varepsilon > 0$, there is $M \in [1, \infty)$, such that if $x \geq M$ then $|f(x) - L| \leq \varepsilon$.

1seq. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} a_n = L \in \mathbb{R} \Leftrightarrow$ for each $\varepsilon > 0$, there exists $M \in \mathbb{N}$, such that if $n \geq M$ then $|a_n - L| \leq \varepsilon$.

Case $L = \infty$.

2fun. $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = \infty \Leftrightarrow$ for each $B \in \mathbb{R}$, there exists $M \in [1, \infty)$, such that if $x \geq M$ then $f(x) \geq B$.

2seq. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} a_n = \infty \Leftrightarrow$ for each $B \in \mathbb{R}$, there exists $M \in \mathbb{N}$, such that if $n \geq M$ then $a_n \geq B$.

Case $L = -\infty$.

3fun. $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = -\infty \Leftrightarrow$ for each $B \in \mathbb{R}$, there exists $M \in [1, \infty)$, such that if $x \geq M$ then $f(x) \leq B$.

3seq. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} a_n = -\infty \Leftrightarrow$ for each $B \in \mathbb{R}$, there exists $M \in \mathbb{N}$, such that if $n \geq M$ then $a_n \leq B$.

Doesn't matter where you start theorem.

Consider two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. (Now think of an integer, like 17, and call him n_0 … so $n_0 = 17$.) Assume that for each $n \geq n_0$, we know that $a_n = b_n$

Theorem: If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = \boxed{}$.

Limits of Functions vs. Limits of Sequences

Consider a function $f : [n_0, \infty) \to \mathbb{R}$ and a sequence $\{a_n\}_n$ such that $a_n = f(n)$ for each $n \geq n_0$.

Theorem 4. Then $\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = L \implies \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} a_n = L.$

Question. What about the reverse implication?

Limits of Functions From Previous Homework

1. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\ln z}{z} = 0$

2. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} z^{1/z} = 1$

3. Let $c > 0$ be a positive constant. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{1/z} = 1$

4a. Let $0 \leq c < 1$ be a constant. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{z} = 0$

4b. Let $c = 1$. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{z} = 1$

4c. Let $c > 1$ be a constant. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{z} = \infty$

5. Let c be a constant. $\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \left(1 + \frac{c}{z}\right)^z = e^c$

Commonly Occurring Limits of Sequences

In the below, you can think of $\lim_{n\to\infty}$ as $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}}$

Also, think of x in below Theorem 5 as a constant c as in above Limits of Functions.

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$

4. $\lim_{n\to\infty} x^{n} = 0 \quad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

For **4**, note if $x \in \mathbb{R}$ then $0 \leq |x^n| = |x|^n \xrightarrow{n\to\infty\text{ , if } |x|<1 \text{ by } \mathbf{4a.}} 0$. Now use Sandwich/Squeeze Theorem.
We will show **6** when we cover the Ratio Test in a few sections.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. As customary, we often shorten $\{a_n\}_{n=1}^{\infty}$ to just $\{a_n\}_n$ or $\{a_n\}$.

Theorems

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

THEOREM 3—The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. hint: $-|a_n| \le a_n \le |a_n|$

Theorem. If $|a_n| \le b_n$ and $\lim_{n\to\infty} |b_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. hint: $-b_n \le -|a_n| \le a_n \le |a_n| \le b_n$

Definitions

1. $\{a_n\}$ is bounded above provided there is a $M \in \mathbb{R}$ such that $a_n \boxed{\le} M$ for all $n \in \mathbb{N}$.
2. $\{a_n\}$ is bounded below provided there is a $M \in \mathbb{R}$ such that $a_n \boxed{\ge} M$ for all $n \in \mathbb{N}$.
3. $\{a_n\}$ is called nondecreasing (or increasing, denoted by $\nearrow$) provided $a_n \boxed{\le} a_{n+1}$ for all $n \in \mathbb{N}$.
4. $\{a_n\}$ is called nonincreasing (or decreasing, denoted by $\searrow$) provided $a_n \boxed{\ge} a_{n+1}$ for all $n \in \mathbb{N}$.
5. $\{a_n\}$ is called monotonic provided $\{a_n\}$ is nondecreasing or nonincreasing.

Theorems

1. Let $\{a_n\}$ be $\nearrow$. Then either
   - $\{a_n\}$ converges (to some finite number)

   or
   - $\{a_n\}$ diverges to ∞.

   Futhermore: $\{a_n\}$ converges (to a finite number) $\iff$ $\{a_n\}$ is bounded above.
2. Let $\{a_n\}$ be $\searrow$. Then either
   - $\{a_n\}$ converges (to some finite number)

   or
   - $\{a_n\}$ diverges to $-\infty$.

   Futhermore: $\{a_n\}$ converges (to a finite number) $\iff$ $\{a_n\}$ is bounded below.