If $y = f(x)$ and $y = g(x)$ are polynomials, then it follows from a theorem in algebra that

$$\frac{f(x)}{g(x)} = P(x) + \frac{F_1(x)}{g(x)} + F_2(x) + \ldots + F_k(x),$$  \hspace{1cm} (1)

where each partial fraction $F_i$ has one of the forms

$$\frac{A}{(px + q)^m} \quad \text{or} \quad \frac{Cx + D}{(ax^2 + bx + c)^n}$$

where

- $m$ and $n$ are nonnegative integers, i.e., $n, m \in \{0, 1, 2, 3, 4, 5, \ldots\}$
- $ax^2 + bx + c$ is irreducible, i.e., it cannot be factored over $\mathbb{R}$, i.e. $b^2 - 4ac < 0$.

Why do we care? Well, if (1) holds then

$$\int \frac{f(x)}{g(x)} \, dx = \int P(x) \, dx + \int [F_1(x) + F_2(x) + \ldots + F_k(x)] \, dx.$$ 

So how to find this decomposition? ....

**First Case:** $[\text{degree of } y = f(x)] < [\text{degree of } y = g(x)]$

In this case, $P(x) = 0$ in (1). Express $y = g(x)$ as a product of

- linear factors $px + q$
- irreducible quadratic factors $ax^2 + bx + c$ (irreducible means that $b^2 - 4ac < 0$).

Collect up the repeated factors so that $g(x)$ is a product of different factors of the form $(px + q)^m$ and $(ax^2 + bx + c)^n$. Then apply the following rules.

**Rule 1:** For each factor of the form $(px + q)^m$ where $m \geq 1$, the decomposition (1) contains a sum of $m$ partial factions of the form

$$\frac{A_1}{(px + q)^1} + \frac{A_2}{(px + q)^2} + \ldots + \frac{A_m}{(px + q)^m}$$

where each $A_i$ is a real number.

**Rule 2:** For each factor of the form $(ax^2 + bx + c)^n$ where $n \geq 1$ and $b^2 - 4ac < 0$, the decomposition (1) contains a sum of $n$ partial factions of the form

$$\frac{A_1x + B_1}{(ax^2 + bx + c)^1} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where the $A_i$'s and $B_i$'s are real number.
Second Case: $\deg y = f(x) \geq \deg y = g(x)$

First do long division to express $\frac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} = \frac{P(x)}{g(x)} + \frac{R(x)}{g(x)} ,$$

How to do this? Well we surely see that

$$\frac{5}{3} = 1\frac{2}{3} = 1 + \frac{2}{3} ;$$

we get this by long division

$$\begin{array}{c}
\frac{1}{3}\sqrt{5} \\
3 \\
2 \\
\end{array}$$

Similarly,

$$\frac{f(x)}{g(x)} = \frac{P(x)}{g(x)} + \frac{R(x)}{g(x)} ,$$

where

$$\begin{array}{c}
P(x) \\
g(x)\sqrt{f(x)} \\
\vdots \\
R(x) \\
\end{array}$$

Now we can apply the First Case to $\frac{R(x)}{g(x)}$ since $\deg y = R(x) < \deg y = g(x)$.

A common mistake. Note that $x^2 = (x-0)^2 = 1x^2 + 0x + 0$ and so $b^2 - 4ac = 0 \not< 0$. So we follow **Rule 1** to see that the partial fraction decomposition of $\frac{1}{x^2}$ is of the form

$$\frac{1}{x^2} = \frac{A}{x^1} + \frac{B}{x^2} .$$

Note that $A = 0$ and $B = 1$. A common mistake is to try to use **Rule 2**, which would give

$$\frac{1}{x^2} = \frac{Ex + F}{x^1} + \frac{Gx + H}{x^2} .$$

This would still lead to the correct answer ($E = F = G = 0$ and $H = 1$) but you have to do LOTS of work to get to it.