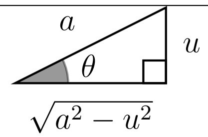
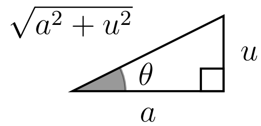
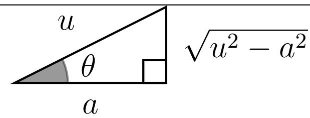
Trig. Substitution is often used when the integrand involves:

$\boxed{a^2 - u^2}$ or $\boxed{a^2 + u^2}$ or $\boxed{u^2 - a^2}$,

Throughout this handout, $\boxed{a > 0}$ is a positive constant.

Logic Reduces Memorization			
If let	**then get**	**Thus if integrand has**	**try letting**
$u = a\sin\theta$ $\frac{u}{a} = \sin\theta$		$a^2 - u^2$	$u = a\sin\theta$
$u = a\tan\theta$ $\frac{u}{a} = \tan\theta$		$a^2 + u^2$	$u = a\tan\theta$
$u = a\sec\theta$ $\frac{u}{a} = \sec\theta$		$u^2 - a^2$	$u = a\sec\theta$

SUMMARY CHART						
	Goal: Find $\int f(u)\, du$	**if** $f(u)$ has	**then** do trig. sub.	**i.e.** do trig. sub.	**need**	& **get**
	↓**recall**↓		**working form**	**reality form**		
(1)	$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$	$a^2 - u^2$	$u = a\sin\theta$	$\theta = \arcsin\left(\frac{u}{a}\right)$	$\left\lvert\frac{u}{a}\right\rvert \leq 1$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ $(\star_1)$
(2)	$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$	$a^2 + u^2$	$u = a\tan\theta$	$\theta = \arctan\left(\frac{u}{a}\right)$	none	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $(\star_2)$
(3)	$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{\lvert u\rvert}{a} + c$	$u^2 - a^2$	$u = a\sec\theta$	$\theta = \text{arcsec}\left(\frac{u}{a}\right)$	$1 \leq \left\lvert\frac{u}{a}\right\rvert$	
(3^+)	if $\sec\theta \geq 0$ (so $u \geq 0$)				$a \leq u$	$0 \leq \theta < \frac{\pi}{2}$ $(\star_{3^+})$
(3^-)	if $\sec\theta \leq 0$ (so $u \leq 0$)				$u \leq -a$	$\frac{\pi}{2} < \theta \leq \pi$ $(\star_{3^-})$

Helpful for when $\sqrt{}$ sign is involved.	
(1)	If the integrand involves $a^2 - u^2$, then let $u = a\sin\theta$ and will have: $\sqrt{a^2-u^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} \overset{(\triangle)}{=} a\lvert\cos\theta\rvert \overset{(\star_1)}{=} a\cos\theta$
(2)	If the integrand involves $a^2 + u^2$, then let $u = a\tan\theta$ and will have: $\sqrt{a^2+u^2} = \sqrt{a^2 + a^2\tan^2\theta} = a\sqrt{1+\tan^2\theta} \overset{(\triangle)}{=} a\lvert\sec\theta\rvert \overset{(\star_2)}{=} a\sec\theta$
(3^+)	If the integrand involves $u^2 - a^2$, then let $u = a\sec\theta$ and, in the case that $a \leq u$, will have: $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} \overset{(\triangle)}{=} a\lvert\tan\theta\rvert \overset{(\star_{3^+})}{=} + a\tan\theta$
(3^-)	If the integrand involves $u^2 - a^2$, then let $u = a\sec\theta$ and, in the case that $u \leq -a$, will have: $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} \overset{(\triangle)}{=} a\lvert\tan\theta\rvert \overset{(\star_{3^-})}{=} - a\tan\theta$
(3)	If book does not provide enough info to determine which (3) to use, then use (3^+).
Self-check: if $(\triangle)$ does not follow from $\cos^2\theta + \sin^2\theta = 1$, then you picked the wrong trig substitution.	

Example 1: Using Trigonometric Substitution, derive the formula

$$\int \frac{dx}{a^2+x^2} \overset{a\geq 0}{=} \frac{1}{a}\tan^{-1}\frac{x}{a} + C\ .$$

Let $a > 0$. Since the integrand involves an expression of the form a^2+x^2, as suggested in the Summary Chart, we try the substitution

$$\boxed{x = a\tan\theta}\ . \qquad \text{(subtitution)}$$

Let's calculate what will be useful for our θ-$d\theta$ substitution:

$$dx =$$

$$a^2 + x^2 =$$

where at $(*)$ we used $1+\tan^2\theta = \sec^2\theta$. Now we are ready to substitute into the original integral:

$$\int \frac{dx}{a^2+x^2} =$$

Example 2: Using Trigonometric Substitution, evaluate the integral

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

Observe that $9-x^2 = (3)^2 + (x)^2$. Since the integrand involves an expression of the form 3^2-x^2, as suggested in the Summary Chart, we try the substitution

$$\boxed{x = \qquad\qquad}\ . \qquad \text{(substitution)}$$

Let's calculate what will be useful for our θ-$d\theta$ substitution:

$$dx =$$

$$\sqrt{9-x^2} \overset{Ⓐ}{=} \sqrt{3^2-x^2} =$$

where at $(*)$ we used $\cos^2\theta + \sin^2\theta = 1$. Now we are ready to substitute into the original integral:

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx =$$

Example 3: Evaluate $\int \frac{dx}{x^2-2x+10}$.

$$\int \frac{dx}{x^2 - 2x + 10} \ = \qquad\qquad + \text{C}$$

Example 4: Evaluate

$$\int_{x=-2}^{x=-\sqrt{2}} \frac{dx}{x^2\sqrt{x^2-1}} \ =$$

Example 5: Evaluate the integral $\int \frac{1}{(4x^2+9)^2}\,dx$. Since the integrand

$$\frac{1}{(4x^2+9)^2} = \frac{1}{\left(\,[2x]^2+3^2\,\right)^2} = \frac{1}{\left(\,u^2+a^2\,\right)^2} \quad \text{, where} \quad u=2x \quad \text{and} \quad a=3 \text{ ,}$$

contains a u^2+a^2, as suggested in the Summary Chart, we try the substitution $u = a\tan\theta$ to get

$$2x = 3\tan\theta \qquad \text{(sub)}$$

$$2\,dx = 3\sec^2\theta\,d\theta$$

$$4x^2+9 \;=\; (2x)^2+3^2 \;=\; (3\tan\theta)^2+3^2 \;=\; 3^2\tan^2\theta+3^2 \;=\; 3^2\left[\tan^2\theta+1\right] \overset{(*)}{=} 3^2\sec^2\theta \text{ ,}$$

where at $(*)$ we used $1+\tan^2\theta = \sec^2\theta$. Now we are ready to substitute into the original integral.

$$\int \frac{dx}{[4x^2+9]^2} = \int \frac{\frac{3}{2}\sec^2\theta\;d\theta}{[3^2\sec^2\theta]^2} = \frac{3}{2}\,\frac{1}{3^4}\int \frac{\sec^2\theta\;d\theta}{\sec^4\theta} = \frac{1}{2\cdot 3^3}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{2\cdot 3^3}\int \cos^2\theta\;d\theta$$

now use a half $\angle$ formula: $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.

$$= \frac{1}{2\cdot 3^3}\,\frac{1}{2}\int (1+\cos(2\theta))\;d\theta \;=\; \frac{1}{2^2\cdot 3^3}\left[\int 1\;d\theta + \int \cos(2\theta)\;d\theta\right]$$

$$= \frac{1}{2^2\cdot 3^3}\left[\int 1\;d\theta + \frac{1}{2}\int \cos(2\theta)\;(2d\theta)\right]$$

$$= \frac{1}{2^2\cdot 3^3}\left[\theta + \frac{1}{2}\sin(\underline{2}\theta)\right] + C$$

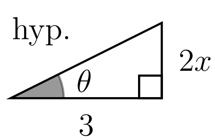
We want our answer in terms of x, not θ. Using $2x = 3\tan\theta$, we can compute (A TRIG FUNCTION)(θ) but we have $\sin(\underline{2}\theta)$. To get rid of that unwanted $\underline{2}$, we use the double $\angle$ formula $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$= \frac{1}{2^2\cdot 3^3}\left[\theta + \frac{1}{2}\,2\sin\theta\,\cos\theta\right] + C$$

$$= \frac{1}{2^2\cdot 3^3}\,\theta \;+\; \frac{1}{2^2\cdot 3^3}\sin\theta\,\cos\theta + C$$

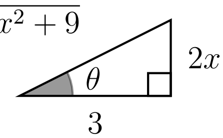
How to find (THE NEEDED TRIG FUNCTION)(θ)? Reference Triangle using original substitution.

$$2x = 3\tan\theta \quad\Rightarrow\quad \boxed{\tan\theta = \frac{2x}{3}} = \frac{\text{opposite}}{\text{adjacent}} \;. \quad \text{Know :}\;\; \sin\theta = \frac{\text{opp.}}{\text{hyp.}} \;\;\&\;\; \cos\theta = \frac{\text{adj.}}{\text{hyp.}}$$



$\Rightarrow$

$$\sqrt{3^2+(2x)^2} = \sqrt{4x^2+9}$$



$$= \frac{1}{2^2\cdot 3^3}\arctan\left(\frac{2x}{3}\right) \;+\; \frac{1}{2^2\cdot 3^3}\cdot\frac{2x}{\sqrt{4x^2+9}}\cdot\frac{3}{\sqrt{4x^2+9}} + C$$

$$= \frac{1}{2^2\cdot 3^3}\arctan\left(\frac{2x}{3}\right) \;+\; \frac{1}{2\cdot 3^2}\cdot\frac{x}{4x^2+9} + C$$

$$= \boxed{\frac{1}{108}\arctan\left(\frac{2x}{3}\right) + \frac{x}{18(4x^2+9)} + C} \;.$$