The book gives lots of *rules/strategies* for Trigonometric Integrals
You are welcome to read them but do NOT memorize.
Use logic instead, as done below!

- Recall the following natural pairings (in terms of derivatives):
$$\sin \leftrightarrow \cos \qquad \tan \leftrightarrow \sec \qquad \cot \leftrightarrow \csc$$
If the integrand does not contain natural pairings, then rewrite it so that it does.
Eg: $\tan^2 x \cos x = \frac{\sin^2 x}{\cos^2 x}\cos x = \frac{\sin^2 x}{\cos x}$ or $\tan^2 x \cos x = \tan^2 x \frac{1}{\sec x} = \frac{\tan^2 x}{\sec x}$.
- Next try a u-du substitution. There will be two naturals choice for u.
  - If the integrand involves sine and/or cosine, then try $u = \cos x$ or $u = \sin x$.
  - If the integrand involves tangent and/or secant, then try $u = \tan x$ or $u = \sec x$.
  - If the integrand involves cotangent and/or cosecant, then try $u = \cot x$ or $u = \csc x$.
- First isolate off your du (box him and don't touch him anymore) and then try to express what's left in terms of only u.

The following will be helpful (and you must memorize them):

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2} \qquad \text{(half-angle } (\tfrac{1}{2}\angle) \text{ formulas)}$$
$$\cos(2x) = \cos^2 x - \sin^2 x \quad \text{and} \quad \sin(2x) = 2\sin x \cos x \qquad \text{(double-angle } (2\angle) \text{ formulas)}$$

$$\cos^2 x + \sin^2 x = 1 \implies \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \implies 1 + \tan^2 x = \sec^2 x$$
$$\cos^2 x + \sin^2 x = 1 \implies \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \implies \cot^2 x + 1 = \csc^2 x$$

►. For integrands of the form: $\int \sin(\alpha x)\cos(\beta x)\, dx$ or $\int \sin(\alpha x)\sin(\beta x)\, dx$ or $\int \cos(\alpha x)\cos(\beta x)\, dx$ where $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$ (see Example 5) use the trig identities (these 3 you do not have to memorize)

$$\begin{aligned}
(1)\ \sin(A)\ \cos(B) &= \tfrac{1}{2}[\sin(A-B) + \sin(A+B)] \\
(2)\ \sin(A)\ \sin(B) &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)] \\
(3)\ \cos(A)\ \cos(B) &= \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]\ .
\end{aligned}$$

Example 4 continued from last page.
Our answers from Example's 4's try 1 and try 2 should be the same ... and they are since:

$$\begin{aligned}
\frac{\sec^4\theta}{4} + K &\overset{Ⓐ}{=} \frac{1}{4}\left(\sec^2\theta\right)^2 + K \\
&\overset{Ⓣ}{=} \frac{1}{4}\left(1+\tan^2\theta\right)^2 + K \\
&\overset{Ⓐ}{=} \frac{1}{4}\left(1 + 2\tan^2\theta + \tan^4\theta\right) + K \\
&\overset{Ⓐ}{=} \frac{\tan^2\theta}{2} + \frac{\tan^4\theta}{4} + \left(K + \frac{1}{4}\right). \qquad \text{So} \quad C = K + \frac{1}{4}.
\end{aligned}$$

Example 5. $\int \sin(2x)\cos(3x)\, dx$. Hint: use (1) above with $A = 2x$, $B = 3x$.

$$\int \sin(2x)\cos(3x)\, dx \underset{(1)}{\overset{\text{above}}{=}} \frac{1}{2}\int [\sin(2x-3x) + \sin(2x+3x)]\, dx \overset{Ⓐ}{=} \frac{1}{2}\int [\sin(-x) + \sin(5x)]\, dx$$
$$= \frac{1}{2}\left[(-1)\int \sin(-1x)\,(-1\,dx)\right] + \frac{1}{2}\left[\left(\frac{1}{5}\right)\int \sin(5x)\ (5\,dx)\right]$$
$$= \frac{\cos(-x)}{2} - \frac{\cos(5x)}{10} + C \underset{\cos(-x)=\cos x}{\overset{\text{recall}}{=}} \frac{\cos x}{2} - \frac{\cos(5x)}{10} + C\ .$$

Example 1. $\int \sin^7 x\,dx$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}}$ or $\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}}$

$\boxed{\text{try 1:} \qquad u = \cos x \qquad \text{so} \qquad du = -\sin x\,dx}$

First box off your du:

$$\int \sin^7 x\,dx = -\int \left[\sin^6 x\right] \boxed{-\sin x\,dx}\,.$$

Then try to express what's left, i.e. $\left[\sin^6 x\right]$, in terms of only u, i.e., in terms of only $\cos x$:

$$\sin^6 x = (\sin x)^6 = \left(\sin^2 x\right)^3 = \left(1-\cos^2 x\right)^3 \,. \qquad ☺ \text{ did it.}$$

So:

$$\begin{aligned}
\int \sin^7 x\,dx &= -\int \left[\sin^6 x\right] \boxed{-\sin x\,dx} \\
&= -\int \left[\left(\sin^2 x\right)^3\right] \boxed{-\sin x\,dx} \\
&= -\int \left[\left(1-\cos^2 x\right)^3\right] \boxed{-\sin x\,dx} \qquad (*_{\text{Ex.1}} \text{ for } u = \cos x)
\end{aligned}$$

now that we have a clever way to look at the integral so can do u-du sub, it's smooth sailing from here on out!

$$\begin{aligned}
&= -\int \left[\left(1-u^2\right)^3\right] \boxed{du} \\
&\overset{Ⓐ}{=} \int \left[\left(u^2-1\right)^3\right] \boxed{du} \\
&\overset{Ⓐ}{=} \int \left(u^6 - 3u^4 + 3u^2 - 1\right)\,du \\
&= \frac{u^7}{7} - \frac{3u^5}{5} + \frac{3u^3}{3} - u + C \\
&= \frac{\cos^7 x}{7} - \frac{3\cos^5 x}{5} + \cos^3 x - \cos x + C.
\end{aligned}$$

$\boxed{\text{try 2:} \qquad u = \sin x \qquad \text{so} \qquad du = \cos x\,dx}$

First box off your du:

$$\int \sin^7 x\,dx = \int [???] \boxed{\text{would want here } \boxed{\cos x\,dx}}\,.$$

The substitution with $u = \sin x$ doesn't work because there is no $\cos x$ in the integrand to help make the needed du. ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹

Example 2. $\int \sin^4(17x)\cos^3(17x)\,dx$

$u = \cos(17x)$ $du = -17\sin(17x)\,dx$	or	$u = \sin(17x)$ $du = 17\cos(17x)\,dx$

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try 1:	$u = \cos(17x)$	so	$du = -17\sin(17x)\ dx$

First box off your du:

$$\int \sin^4(17x)\cos^3(17x)\,dx = \frac{-1}{17}\int \left[\cos^3(17x)\sin^3(17x)\right]\boxed{-17\sin(17x)\,dx}\ .$$

Then try to express what's left, i.e. $\left[\cos^3(17x)\sin^3(17x)\right]$, in terms of only u, i.e., in terms of only $\cos(17x)$:

$$\cos^3(17x)\sin^3(17x) = \cos^3(17x)\sin^2(17x)\sin(17x) = \cos^3(17x)\ \left(1-\cos^2(17x)\right)\ \sin(17x)\ .$$

Cannot do, there is an "extra sin" ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹ ☹

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try 2:	$u = \sin(17x)$	so	$du = 17\cos(17x)\ dx$

First box off your du:

$$\int \sin^4(17x)\cos^3(17x)\,dx = \frac{1}{17}\int \left[\sin^4(17x)\cos^2(17x)\right]\boxed{17\cos(17x)\,dx}\ .$$

Then try to express what's left, i.e. $\left[\sin^4(17x)\cos^2(17x)\right]$, in terms of only u, i.e., in terms of only $\sin(17x)$:

$$\sin^4(17x)\cos^2(17x) = \sin^4(17x)\left(1-\sin^2(17x)\right)$$ ☺ did it.

So:

$$\begin{aligned}
&\int \sin^4(17x)\cos^3(17x)\,dx \\
&= \frac{1}{17}\int \left[\sin^4(17x)\cos^2(17x)\right]\boxed{17\cos(17x)\,dx} \\
&= \frac{1}{17}\int \left[\sin^4(17x)\ \left(1-\sin^2(17x)\right)\right]\boxed{17\cos(17x)\,dx} \qquad (*_{\text{Ex.2}} \text{ for } u=\sin(17x)) \\
&= \frac{1}{17}\int \left[u^4\left(1-u^2\right)\right]\boxed{du} \\
&\overset{\text{(A)}}{=} \frac{1}{17}\int (u^4-u^6)\,du \\
&= \frac{1}{17}\left(\frac{u^5}{5}-\frac{u^7}{7}\right)+C = \frac{1}{17}\left(\frac{\sin^5(17x)}{5}-\frac{\sin^7(17x)}{7}\right)+C\ .
\end{aligned}$$

Example 3. $\int \sin^4 x\,dx$. u-du sub does not work (why? e.g.: $\int \sin^4 x\,dx \overset{u=\cos x}{=} -\int \boxed{\sin^3 x}\,[-\sin x\,dx]$).
For $\int \sin^n x \cos^m x\,dx$, with BOTH $m, n \in \{0, 2, 4, 6, \ldots\}$, use the half-angle formulas.

$$\int \sin^4 x\,dx \overset{Ⓐ}{=} \int [\sin^2 x]^2\,dx \overset{(\frac{1}{2}\angle)}{=} \int \left[\frac{1-\cos(2x)}{2}\right]^2 dx \overset{Ⓐ}{=} \frac{1}{4}\int [1 - 2\cos(2x) + \cos^2(2x)]\,dx$$

$$\overset{(\frac{1}{2}\angle)}{=} \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] dx \overset{Ⓐ}{=} \frac{1}{4}\int \left[\left(1+\tfrac{1}{2}\right) - 2\cos(2x) + \tfrac{1}{2}\cos(4x)\right] dx$$

$$= \frac{1}{4}\int \frac{3}{2}\,dx - \frac{1}{4}\int 2\cos(2x)\,dx + \frac{1}{4}\cdot\frac{1}{2}\int \cos(4x)\,dx$$

$$= \frac{3}{8}\int dx - \frac{1}{4}\cdot\int \cos(2x)\;[2dx] + \left(\frac{1}{4}\cdot\frac{1}{2}\right)\cdot\left(\frac{1}{4}\right)\int \cos(4x)\;[4dx]$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

to here is fine now but in trig-sub we'll want all angles to be $(1x)$ so let's do it . . . note: $\sin(4x) = \sin(2\,(2x))$

$$\overset{(2\angle)}{=} \frac{3}{8}x - \frac{1}{4}(2\sin x\cos x) + \frac{1}{32}[(2)\sin(2x)\cos(2x)] + C$$

$$\overset{(2\angle)}{=} \frac{3}{8}x - \frac{1}{2}(\sin x\cdot\cos x) + \frac{1}{32}\left[(2)\,(2\sin x\cdot\cos x)\left(\cos^2 x - \sin^2 x\right)\right] + C.$$

Good enough for now.

Example 4. $\int \tan\theta\sec^4\theta\,d\theta$ so: $\boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{array}}$ or $\boxed{\begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta\,d\theta \end{array}}$

$\boxed{\text{try 1:} \quad u = \tan\theta \quad \text{so} \quad du = \sec^2\theta\,d\theta}$

First box off your du:

$$\int \tan\theta\sec^4\theta\,d\theta = \int \left[\tan\theta\sec^2\theta\right]\boxed{\sec^2\theta\,d\theta}\,.$$

Then try to express what's left, i.e. $[\tan\theta\sec^2\theta]$, in terms of only u, i.e. in terms of only $\tan\theta$:

$$\left[\tan\theta\sec^2\theta\right] = \tan\theta\left(1+\tan^2\theta\right) \qquad ☺ \text{ did it.}$$

So for $\underline{\underline{u = \tan\theta}}$,

$$\int \tan\theta\sec^4\theta\,d\theta = \int \left[\tan\theta\sec^2\theta\right]\boxed{\sec^2\theta\,d\theta}$$

$$= \int \left[\tan\theta\left(1+\tan^2\theta\right)\right]\boxed{\sec^2\theta\,d\theta}$$

$$= \int \left[u(1+u^2)\right]\boxed{du} \overset{Ⓐ}{=} \int (u+u^3)\,du = \frac{u^2}{2} + \frac{u^4}{4} + C = \frac{\tan^2\theta}{2} + \frac{\tan^4\theta}{4} + C.$$

$\boxed{\text{try 2:} \quad u = \sec\theta \quad \text{so} \quad du = \sec\theta\tan\theta\,d\theta}$

First box off your du:

$$\int \tan\theta\sec^4\theta\,d\theta = \int \left[\sec^3\theta\right]\boxed{\sec\theta\tan\theta\,d\theta}\,.$$

Then try to express what's left, i.e. $[\sec^3\theta]$, in terms of only u, i.e. in terms of only $\sec\theta$. ☺$_{\text{easy}}$
So for $\underline{\underline{u = \sec\theta}}$

$$\int \tan\theta\sec^4\theta\,d\theta = \int \left[\sec^3\theta\right]\boxed{\sec\theta\tan\theta\,d\theta} = \int \left[u^3\right]\boxed{du} = \frac{u^4}{4} + K = \frac{\sec^4\theta}{4} + K.$$

All done. See bottom of first page for a comment on this problem.