Setup. Given $y = f(x)$ and x_0 in an interval I. Know the derivatives of $y = f^{(n)}(x)$ exist for each $x \in I$ and for each $n \in \mathbb{N}$.

The *N^{th}-order Taylor polynomial* for $y = f(x)$ at x_0 is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N , \qquad \text{(open form)}$$

which can also be written in open form as (recall that $0! = 1$)

$$p_N(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N ,$$

which can also be written in closed form, by using sigma notation, as

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n . \qquad \text{(closed form)}$$

So $y = p_N(x)$ is a polynomial of degree at most N and it has the form

$$p_N(x) = \sum_{n=0}^{N} c_n (x - x_0)^n \quad \text{where the } n^{\text{th}} \text{ Taylor coefficients} \quad c_n = \frac{f^{(n)}(x_0)}{n!} \text{ are constants.}$$

The c_n is choosen so that $f^{(n)}(x_0) = (p_n)^{(n)}(x_0)$. Knowing $y = P_N(x)$, how can you find $y = P_{N+1}(x)$?

$$p_{N+1}(x) = \sum_{n=0}^{N+1} c_n (x-x_0)^n = \left[\sum_{n=0}^{N} c_n (x - x_0)^n\right] + \left(c_{N+1} (x - x_0)^{N+1}\right) = P_N(x) + c_{N+1} (x-x_0)^{N+1}.$$

The *Taylor series* for $y = f(x)$ at x_0 is the power series:

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \text{(open form)}$$

which can also be written as

$$P_\infty(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots \quad \hookleftarrow \text{the sum keeps on going and going.}$$

The Taylor series can also be written in closed form, by using sigma notation, as

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n . \qquad \text{(closed form)}$$

The *Maclaurin polynomial/series* for $y = f(x)$ is just the Taylor polynomial/series for $y = f(x)$ at $x_0 = 0$.

Example 1. Given the function $f(x) = \frac{1}{1-x}$ with center at $x_0 = 0$.

Helpful Table for Example 1			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(0)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(0)}{n!}$
0	$(1-x)^{-1}$	$(1-0)^{-1} = 1$	$\frac{1}{0!} = \frac{1}{1} \overset{\text{note}}{=} \frac{0!}{0!} = 1$
1	$-(1-x)^{-2}(-1) = (1-x)^{-2}$	$(1-0)^{-2} = 1$	$\frac{1}{1!} = \frac{1!}{1!} = 1$
2	$-2(1-x)^{-3}(-1) = 2(1-x)^{-3}$	$2(1-0)^{-3} = 2$	$\frac{2}{2!} = \frac{2!}{2!} = 1$
3	$2(^-3)(1-x)^{-4}(-1) = 3!(1-x)^{-4}$	$3!(1-0)^{-4} = 3!$	$\frac{3!}{3!} = 1$
4	$3!(^-4)(1-x)^{-5}(-1) = 4!(1-x)^{-5}$	$4!(1-0)^{-5} = 4!$	$\frac{4!}{4!} = 1$
5	$4!(^-5)(1-x)^{-6}(-1) = 5!(1-x)^{-6}$	$5!(1-0)^{-6} = 5!$	$\frac{5!}{5!} = 1$
6	$5!(^-6)(1-x)^{-7}(-1) = 6!(1-x)^{-7}$	$6!(1-0)^{-7} = 6!$	$\frac{6!}{6!} = 1$
$\vdots$			
n			

1.1. Find the N^{th}-order Taylor polynomial of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$ for $N = 0, 1, 2, 3, 4$.
Do not use $\sum$-sign.

$p_0(x) =$ ______________________ , cleaned up, $p_0(x) =$ ______________________

$p_1(x) =$ ______________________ , cleaned up, $p_1(x) =$ ______________________

$p_2(x) =$ ______________________ , cleaned up, $p_2(x) =$ ______________________

$p_3(x) =$ ______________________ , cleaned up, $p_3(x) =$ ______________________

$p_4(x) =$ ______________________ , cleaned up, $p_4(x) =$ ______________________

1.2. Express the 4^{th}-order Taylor polynomial of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$ in closed form (i.e., with $\sum$-sign).

$p_4(x) =$ [ ] , cleaned up, $p_4(x) =$ [ ]

1.3. Fill in the last row in the Helpful Table for Example 1.

1.4. What is the n^{th} Taylor coefficient of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$?

$c_n =$ [ ] for $n = 0, 1, 2, 3, 4, 5, \ldots$.

1.5. Express the N^{th}-order Taylor polynomial of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$ in closed form (i.e., use $\sum$-sign).

$p_N(x) =$ [ ] , cleaned up, $p_N(x) =$ [ ]

1.6. Find the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$. Use closed form (i.e., use $\sum$-sign).

$P_\infty(x) =$ [ ] , cleaned up, $P_\infty(x) =$ [ ]

1.7. Express the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$ in open form (i.e., use $\ldots$-sign).

$P_\infty(x) =$ ____

or, by including the general term, we can also express as

$P_\infty(x) =$ ____

1.8. Find the interval of convergence of the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$.

►. Aside. Let's think about this example some more. We were given the center $x_0 = 0$ and the function

$$f(x) = \frac{1}{1-x} .$$

Then we have computed the Taylor series P_∞ as

$$P_\infty(x) = \sum_{n=0}^{\infty} x^n . \tag{1.1}$$

and showed that power series series in (1.1) converges when $|x| < 1$.

Recall that for the Geometric Series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{, which is valid when } |r| < 1,$$

which we showed some time ago by considering the $s_n - rs_n$. Replacing r by x we get

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{, which is valid when } |x| < 1,$$

which gives that the function $y = \frac{1}{1-x}$ has a power series representation.

Theorem 1.1. *Power series vs. Taylor series.*
If a function $y = f(x)$ has a power series representation about center $x = x_0$ with radius of conv. $R > 0$, then that power series representation is the Taylor series for $y = f(x)$ about center $x = x_0$.

I.e. (just going to repharse ...)

If for some $R > 0$

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \text{, which is valid when } |x - x_0| < R \tag{1.2}$$

then the c_n's must satisfy

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

and so the power series in (1.2) is the Taylor series of $y = f(x)$ about the center $x = x_0$.

To see why this Theorem is true, see the handout *Power Series vs. Taylor Series.*

Warning.
It is possible for a function not to be equal to it's Taylor series, which as radius of convergence $R > 0$.
For such a function f, all its derivaties $f^{(n)}$ exist so you can (formally) write down it's Taylor series.
But it's Taylor series does not converge to the function f.
For an example of such a function, see the handout *Power Series vs. Taylor Series.*
In this example the center is $x_0 = 0$ and the Taylor series is $P_\infty(x) = 0$ for each $x \in \mathbb{R}$.
But $f(x) = 0$ only when x is ________.

Example 2. Given the function $f(x) = \sin x$ with center at $x_0 = \pi$.

Helpful Table for Example 1					
k	$2k+1$	n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(\pi)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(\pi)}{n!}$
		0	$\sin x$	$\sin \pi = 0$	$\frac{0}{0!} = \frac{0}{1} = 0$
		1	$\cos x$	$\cos \pi = {}^{-}1$	$\frac{-1}{1!} = -\frac{1}{1} = -1$
		2	$-\sin x$	$-\sin \pi = 0$	$\frac{0}{2!} = 0$
		3	$-\cos x$	$-\cos \pi = -({}^{-}1) = 1$	$\frac{1}{3!} = +\frac{1}{3!}$
		4	$\sin x$	$\sin \pi = 0$	$\frac{0}{4!} = 0$
		5	$\cos x$	$\cos \pi = {}^{-}1$	$\frac{-1}{5!} = -\frac{1}{5!}$
		6	$-\sin x$	$-\sin \pi = 0$	$\frac{0}{6!} = 0$
		7	$-\cos x$	$-\cos \pi = 1$	$\frac{1}{7!} = +\frac{1}{7!}$

► Note $f^{(4)}(x) = f^{(0)}(x)$ so derivatives repeat/cycle in sets of 4.

2.1. Find the N^{th}-order Taylor polynomial of $f(x) = \sin x$ centered at $x_0 = \pi$ for $N = 0, 1, \ldots, 8$ and $N = 11$. Do not use $\sum$-sign.

$p_0(x) =$ ____________________ , cleaned up, $p_0(x) =$ ____________

$p_1(x) =$ ____________________ , cleaned up, $p_1(x) =$ ____________

$p_2(x) =$ ____________________ , cleaned up, $p_2(x) =$ ____________

Now let's just do the cleaned up version from the onset.

$p_3(x) =$ ______________________________

$p_4(x) =$ ______________________________

$p_5(x) =$ ______________________________

$p_6(x) =$ ______________________________

$p_7(x) =$ ______________________________

$p_8(x) =$ ______________________________

$p_{11}(x) =$ ______________________________

► Why do we use the term N^{th} order Taylor polynomial instead of N^{th} degree Taylor polynomial?

2.2. Express the Taylor series of $f(x) = \sin x$ centered at $x_0 = \pi$ in open form (i.e., use $\ldots$-sign).

$P_\infty(x) =$ ______________________________

2.3. Find the Taylor series of $f(x) = \sin x$ centered at $x_0 = \pi$. Use closed form (i.e., use $\sum$-sign).

Hint: $(-1)^{\text{an even number}} =$ ________ while $(-1)^{\text{an odd number}} =$ ________ .

$P_\infty(x) =$ [ ] , or, $P_\infty(x) =$ [ ]

2.4. Find the interval of convergence and radius of conv. of the Taylor series of $f(x) = \sin x$ centered at $x_0 = \pi$.

2.5. Time to recall one of the Commonly Used Sequences. Let $x \in \mathbb{R}$.

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ is absolutely convergent (hint: ratio test). Conclude that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

2.6. Show that $f(x) = \sin x$ is equal to it's Taylor series centered at $x_0 = \pi$ for each $x \in \mathbb{R}$. I.e., show that

$$\sin x = \boxed{} \quad \text{for each} \quad x \in \mathbb{R}. \tag{2.5}$$

Example 3. Given the function $f(x) = \ln(1+x)$ with center $x_0 = 0$.

Helpful Table for Example 3			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(0)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(0)}{n!}$
0	$\ln(1+x)$	$\ln(1+0) = 0$	$\frac{0}{0!} = \frac{0}{1} = 0$
1	$(1+x)^{-1}$	$(1+0)^{-1} = +1$	$\frac{1}{1!} = +1$
2	$-(1+x)^{-2}$	$-(1+0)^{-2} = -1$	$\frac{-1}{2!} = -\frac{1}{2}$
3	${}^{+}2(1+x)^{-3}$	${}^{+}2(1+0)^{-3} = +2$	$\frac{2}{3!} = +\frac{1}{3}$
4	${}^{-}3!(1+x)^{-4}$	${}^{-}3!(1+0)^{-4} = -3!$	$\frac{-3!}{4!} = -\frac{1}{4}$
5	${}^{+}4!(1+x)^{-5}$	${}^{+}4!(1+0)^{-5} = +4!$	$\frac{4!}{5!} = +\frac{1}{5}$
6	${}^{-}5!(1+x)^{-6}$	${}^{-}5!(1+0)^{-6} = -5!$	$\frac{-5!}{6!} = -\frac{1}{6}$
$\vdots$	do next line later in ≈ 3.2		
n	when $n \geq 1$: $(-1)^{n-1}(n-1)!(1+x)^{-n}$	$(-1)^{n-1}(n-1)!$	$\frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$

3.1. Find the N^{th}-order Maclaurin polynomial of $f(x) = \ln(1+x)$ for $N = 0, 1, 2, 3, 4$ and $N = 7$.

Do not use $\sum$-sign.

$p_0(x) =$ ________________ , cleaned up, $p_0(x) =$ ________________

$p_1(x) =$ ________________ , cleaned up, $p_1(x) =$ ________________

$p_2(x) =$ ________________ , cleaned up, $p_2(x) =$ ________________

Now let's just do the cleaned up version from the onset.

$p_3(x) =$ ________________________________

$p_4(x) =$ ________________________________

$p_7(x) =$ ________________________________

3.2. Fill in the last row in the Helpful Table for Example 3.

3.3. What is the n^{th} Maclaurin coefficient of $f(x) = \ln(1+x)$?

$c_n =$ [ ] for $n = 1, 2, 3, 4, 5, \ldots$ and $c_0 =$ [ ]

3.4. Express the N^{th}-order Maclaurin polynomial of $f(x) = \ln(1+x)$ in closed form (i.e., use $\sum$-sign).
When $N = 0$ we get $p_0(x) = 0$. And when $N = 1, 2, 3, 4, \ldots$, we have

$p_N(x) =$ [ ] , cleaned up, $p_N(x) =$ [ ]

3.5. Find the Maclaurin series of $f(x) = \ln(1+x)$ Use closed form (i.e., use $\sum$-sign).

$P_\infty(x) =$ [ ] , cleaned up, $P_\infty(x) =$ [ ]

3.6. Remark. If we apply method of power series (the ratio/root test then check endpoints) then we would see the interval of convergence of the Maclaurin seris for $f(x) = \ln(1+x)$ is $(-1, 1]$.
You should do this after class!

►. Fact. The function $f(x) = \ln(1+x)$ is equal to it's Maclaurin series on the interval $x \in (-1, 1]$. I.e.,

$\ln(1+x) =$ [ ] when $x \in (-1, 1]$

Pretty cool but hard to show for $x \in \left(-1, -\frac{1}{2}\right)$. So we'll just show it in the easier case that $x \in \left[-\frac{1}{4}, \frac{5}{8}\right]$.

3.7. Show that on the interval $\left[-\frac{1}{4}, \frac{5}{8}\right]$, the function $f(x) = \ln(1+x)$ is equal to it's Maclaurin series.
I.e., show

$\ln(1+x) =$ [ ] when $-\dfrac{1}{4} \leq x \leq \dfrac{5}{8}$ (3.7)