Read this handout thoroughly and then do Homeworks: 2, 4, and 6.

Let's consider a function

$$y = f(x)$$

and fix a point x_0 in the domain of $y = f(x)$. So the graph of $y = f(x)$ goes through the point

$$(x_0, f(x_0)) \ .$$

The equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is found by:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - f(x_0) &= f'(x_0)(x - x_0) \\ y &= f(x_0) + f'(x_0)(x - x_0) \ . \end{aligned}$$

So the equation $y = p_1(x)$ of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0) \ . \tag{1}$$

Recall that the function $y = f(x)$ can be approximated *locally* near x_0 by this tangent line

$$y = p_1(x) \ .$$

In other words, if x is close to x_0 then the value $f(x)$ is close to the value $p_1(x)$,
that is, if $x \approx x_0$ then $f(x) \approx p_1(x)$.

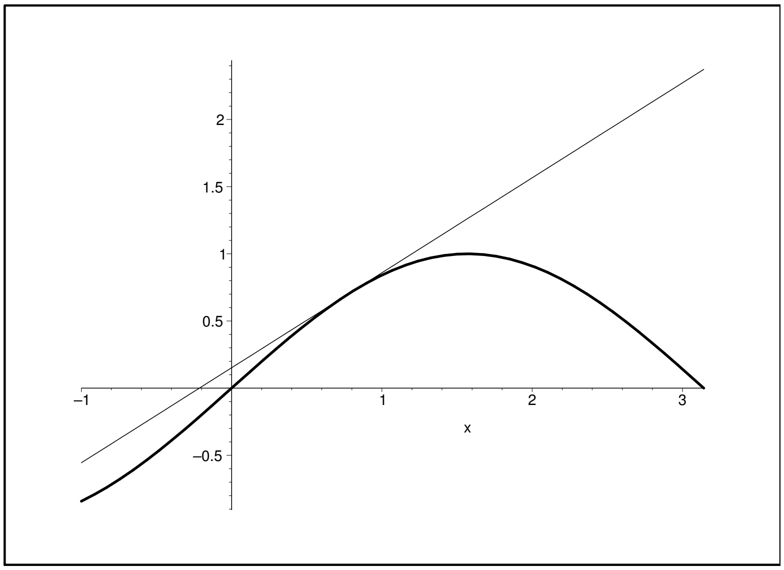
To better see what is going on pictorially/graphically, let's do an example.

Example 1. Near $x_0 = \frac{\pi}{4}$, the function $f(x) = \sin(x)$ can be approximated by the line

$$y = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$\boxed{p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) \ .}$$

Let's graph the function $y = f(x)$ and the line $y = p_1(x)$ on the same grid.



Graphs for Example 1

Homework 2. Find the equation $y = p_1(x)$ of the tangent line to the function $f(x) = \frac{1}{x}$ at the point $x_0 = 2$. Express your answer in the form $p_1(x) = d + m(x - 2)$ for some constants d & m.

Surely you easily knocked out Homework 2. Let's do it together, using ideas that will be helpful in problems-to-come. As we figured out in (1), the equation $y = p_1(x)$ of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0) \; .$$

Let's make a table of what is needed here. We will introduce some notation which will come in handy later.

Helpful Table for Homework 2		
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(2)$
0	$f^{(0)}(x) \overset{\text{def}}{=} f(x) = x^{-1}$	$f^{(0)}(2) = \frac{1}{2}$
1	$f^{(1)}(x) \overset{\text{def}}{=} f'(x) = -\, x^{-2}$	$f^{(1)}(2) = -\frac{1}{4}$

Using Helpful Table for Homework 2 and the equation (1), we get the following.

$$\boxed{p_1(x) = \frac{1}{2} + \frac{-1}{4}(x - 2) \; .} \tag{2}$$

Note that (2) is the the reqested form $p_1(x) = d + m(x - 2)$ where the constants $d = \frac{1}{2}$ and $m = \frac{-1}{4}$ so WE ARE DONE. It's not to hard to put our solution into the form $y = mx + b$ so let's just do it for fun:

$$p_1(x) = \frac{1}{2} - \frac{x}{4} + \frac{1}{2} \qquad \text{so} \qquad \boxed{y = \frac{-1}{4}\, x + 1 \; .}$$

Note that this tangent line approximation works well because the tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$ is *the only line with slope* $f'(x_0)$ *passing through the point* $(x_0, f(x_0))$. We can generalize this to second degree approximations by finding a parabola $y = ax^2 + bx + c$ passing through the point $(x_0, f(x_0))$ with the same slope (first derivative) as $y = f(x)$ at x_0 and the same second derivative as $y = f(x)$ at x_0. We will illustrate how to find such a parabola in Example 3.

Example 3. Consider $f(x) = e^{-(x-1)}$ at $x_0 = 1$. We want to find a parabola $y = p_2(x)$ so that:

0. $p_2(1) = f(1)$ (so $y = f(x)$ and $y = p_2(x)$ both pass thru the same point $(1, f(1))$)
1. $p_2'(1) = f'(1)$ (so $y = f(x)$ and $y = p_2(x)$ have the same first derivative at $x_0 = 1$)
2. $p_2''(1) = f''(1)$ (so $y = f(x)$ and $y = p_2(x)$ have the same second derivative at $x_0 = 1$) .

Such a parabola as we want looks like $p_2(x) = ax^2 + bx + c$ for some constants a, b, c with $a \neq 0$; as you will see, it's better to view as

$$p_2(x) = c_0 + c_1(x - 1) + c_2(x - 1)^2$$

for some carefully chosen **constants** c_0, c_1, c_2 with $c_2 \neq 0$. We just have to figure out how to carefully chose these constants c_0, c_1, c_2. Let's see, we do have/know

$$\begin{aligned} p_2(x) &= c_0 + c_1(x-1) + c_2(x-1)^2 && \text{and} && f(x) = e^{-(x-1)} \\ p_2'(x) &= c_1 + 2c_2(x-1) && \text{and} && f'(x) = -e^{-(x-1)} \\ p_2''(x) &= 2c_2 && \text{and} && f''(x) = +e^{-(x-1)} \end{aligned}$$

and evaluating these at $x_0 = 1$ gives us

$$\begin{aligned} p_2(1) &= c_0 && \text{and} && f(1) = e^{-(0)} = 1 \\ p_2'(1) &= c_1 && \text{and} && f'(1) = -e^{-(0)} = -1 \\ p_2''(1) &= 2c_2 && \text{and} && f''(1) = +e^{-(0)} = 1 \end{aligned}$$

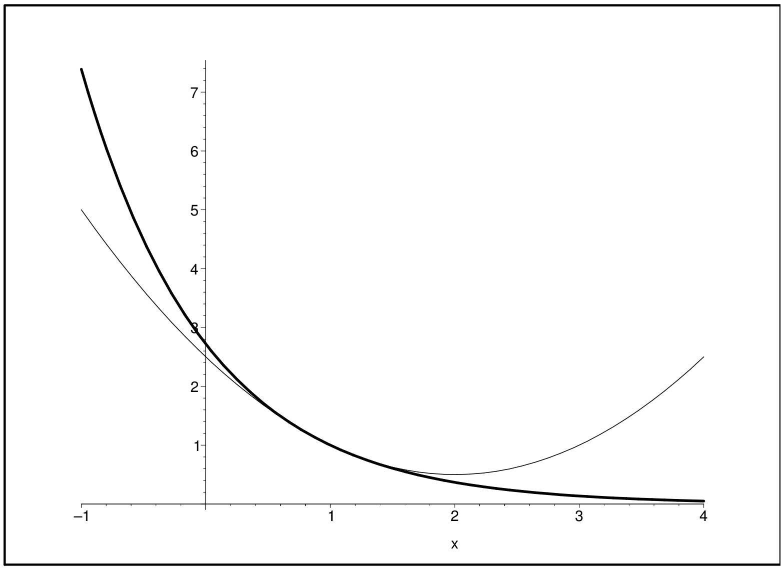
and so

$$\begin{aligned} p_2(1) &= f(1) && \Leftrightarrow && c_0 = 1 \\ p_2'(1) &= f'(1) && \Leftrightarrow && c_1 = -1 \\ p_2''(1) &= f''(1) && \Leftrightarrow && 2c_2 = 1 && \Leftrightarrow && c_2 = \tfrac{1}{2} \; . \end{aligned}$$

So our parabola is

$$\boxed{p_2(x) = 1 - (x-1) + \frac{1}{2}(x-1)^2.}$$

This polynomial is *the second order Taylor polynomial* of $y = e^{-(x-1)}$ centered at $x_0 = 1$. Notice that close to x=1 this parabola approximates the function rather well.



Graphs for Example 3

Using Example 3 as a model we can give a general form of the second order Taylor polynomial for $y = f(x)$ at x_0, that is the parabola

$$p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2$$

where we want to find the **constants** c_0, c_1, c_2 to make the derivatives of $y = f(x)$ and $y = p_2(x)$ match up at $x = x_0$. We have

$$\begin{aligned} p_2(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)^2 && \Longrightarrow && p_2(x_0) = c_0 \\ p_2'(x) &= c_1 + 2c_2(x - x_0) && \Longrightarrow && p_2'(x_0) = c_1 \\ p_2''(x) &= 2c_2 && \Longrightarrow && p_2''(x_0) = 2c_2 \end{aligned}$$

and so

$$\begin{aligned} p_2(x_0) &= f(x_0) && \Longleftrightarrow && c_0 = f(x_0) \\ p_2'(x_0) &= f'(x_0) && \Longleftrightarrow && c_1 = f'(x_0) \\ p_2''(x_0) &= f''(x_0) && \Longleftrightarrow && 2c_2 = f''(x_0) && \Longleftrightarrow && c_2 = \tfrac{f''(x_0)}{2} \; . \end{aligned}$$

So our parablola is

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \; . \tag{3}$$

Compare the function $y = p_1(x)$ in formula (1) with the function $y = p_2(x)$ in formula (3). *Starting to see a pattern?*

Homework 4. Find the second order Taylor polynomial $y = p_2(x)$ for $f(x) = \frac{1}{x}$ at $x_0 = -2$. First fill in the Helpful Table for Homework 4. Then express your answer in the form

$$p_2(x) = c_0 + c_1(x -^{-} 2) + c_2(x -^{-} 2)^2 \quad \text{or} \quad p_2(x) = c_0 + c_1(x + 2) + c_2(x + 2)^2$$

for some constants c_0, c_1, c_2.

Helpful Table for Homework 4			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(-2)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(-2)}{n!}$
0	$f^{(0)}(x) \overset{\text{def}}{=} f(x) = x^{-1}$		
1			
2			

Soln: $p_2(x) =$ ______ .

Higher order Taylor polynomials are found in the same way. For example, the *third order Taylor polynomial* for a function $y = f(x)$ centered at x_0 is

$$p_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 .$$

Big Definition. The *N^{th}-order Taylor polynomial* for $y = f(x)$ at x_0 is:

$$p_N(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N ,$$

which can also be written as (recall that $0! = 1$)

$$p_N(x) = \frac{f^{(0)}(x_0)}{0!} + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x - x_0)^N . \qquad \text{(N - open form)}$$

Formula (N - open form) is in open form. It can also be written in closed form, by using sigma notation, as

$$p_N(x) \;=\; \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n . \qquad \text{(N- closed form)}$$

So $y = p_N(x)$ is a polynomial of degree at most N and it has the form

$$p_N(x) \;=\; \sum_{n=0}^{N} c_n\,(x - x_0)^n$$

where the c_n's

$$c_n \;=\; \frac{f^{(n)}(x_0)}{n!}$$

are specially chosen so that

$$\begin{aligned}
p_N(x_0) &= f(x_0) \\
p_N^{(1)}(x_0) &= f^{(1)}(x_0) \\
p_N^{(2)}(x_0) &= f^{(2)}(x_0) \\
&\vdots \\
p_N^{(N)}(x_0) &= f^{(N)}(x_0) .
\end{aligned}$$

The constant c_n is called the n^{th} Taylor coefficient of $y = f(x)$ about x_0.
The *N^{th}-order Maclaurin polynomial* for $y = f(x)$ is just
the N^{th}-order Taylor polynomial for $y = f(x)$ at $x_0 = 0$ and so it is

$$p_N(x) \;=\; \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\, x^n \;.$$

□

Example 5. Consider the function

$$f(x) = \sin(3x)$$

near (centered) the point $x_0 = 0$. Let's graph, on the same grid, $y = f(x)$ along with its Maclaurin polynomial $y = p_N(x)$ for $N = 1, 3, 5, 7, 9, 11, 13$.

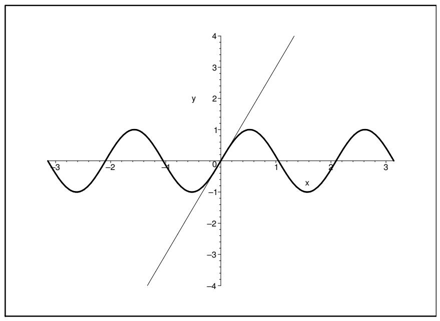


FIGURE 1. $y = \sin(3x)$ along with its first order Maclaurin Polynomial $y = p_1(x)$

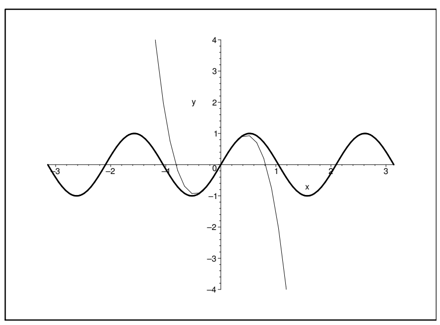


FIGURE 3. $y = \sin(3x)$ along with its third order Maclaurin Polynomial $y = p_3(x)$

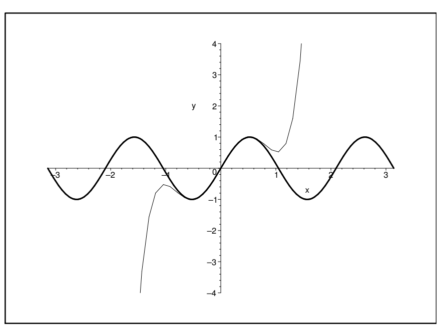


FIGURE 5. $y = \sin(3x)$ along with its fifth order Maclaurin Polynomial $y = p_5(x)$

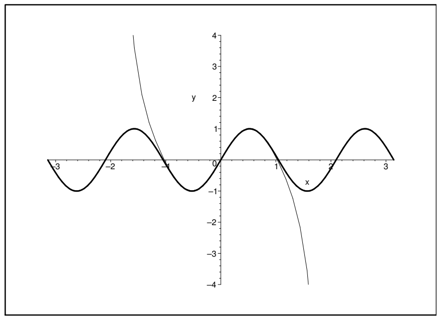


FIGURE 7. $y = \sin(3x)$ along with its 7^{th} order Maclaurin Polynomial $y = p_7(x)$

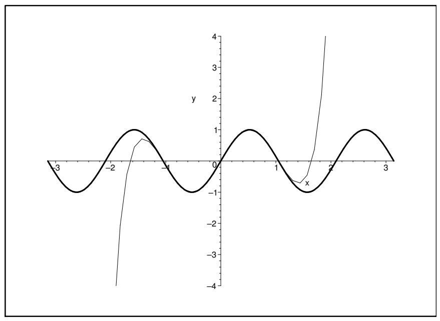


FIGURE 9. $y = \sin(3x)$ along with its 9^{th} order Maclaurin Polynomial $y = p_9(x)$

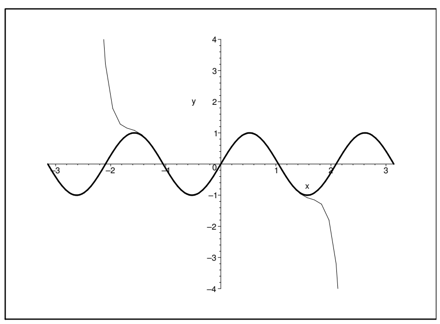


FIGURE 11. $y = \sin(3x)$ along with its 11^{th} order Maclaurin Polynomial $y = p_{11}(x)$

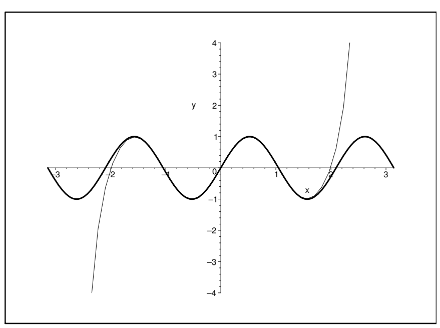


FIGURE 13. $y = \sin(3x)$ along with its 13^{th} order Maclaurin Polynomial $y = p_{13}(x)$

Notice that as N increases the approximation of $y = \sin(3x)$ by $y = p_N(x)$ gets better and better, even over a wider and wider interval around the center $x_0 = 0$. So for a fixed x the approximation of $y = f(x)$ by $y = p_N(x)$ becomes more accurate as N gets bigger.

Homework 6. For the function $f(x) = \sin(3x)$ from Example 5, find the Maclaurin polynomials:

$$y = p_1(x),\ y = p_3(x),\ y = p_5(x),\ y = p_7(x),\ y = p_9(x),\ y = p_{11}(x),\text{ and } y = p_{13}(x)\ .$$

First fill out the Helpful Table and then indicate the Maclaurin polynomials in the Solution Table. We are looking for patterns so you may leave/express, e.g., 3^5 as just 3^5 rather than 243 and 5! as just 5! rather than 120; in short, you do not need a calculator.

Helpful Table for Homework 6			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(0)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(0)}{n!}$
0	$\sin(3x) \overset{\text{note}}{=} +3^0 \sin(3x)$	0	0
1	$3\cos(3x) \overset{\text{note}}{=} +3^1 \cos(3x)$	$+3^1$	$+\frac{3^1}{1!} = +3$
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			

Solution Table for Homework 6	
n	$y = p_n(x)$
1	
3	
5	$p_5(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5$
7	
9	
11	
13	

Just to think about. Take another look at Homework 6. Do you notice any pattern in the Taylor coefficients? Why did we only use odd-order Taylor polynomials?

Bonus problem. In Homework 6, what is the 4^{th}-order Maclaurin polynomial?

Soln: $p_4(x)\ =$ ______ .