exEB. Mr. Energizer Bunny is 1 mile from 5points and he starts hopping (and hopping and hopping).

He hops $\frac{1}{2}$ mile on the first hop and then
on the next hop he hops half as far as he hopped on the previous hop. (1EB)

Does he ever get to 5points in a finite amont of time? If not, how far (assuming he goes on forever) does he hop?

Soln. For $n \in \mathbb{N}$, let

$$\begin{aligned} a_n &= \text{amount he hops on the } n^{\text{th}} \text{ hop} \\ s_n &= \text{total amount he has hopped after } n \text{ hops.} \end{aligned}$$

We want to examine the sequence $\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3, s_4, \ldots\}$. From (1EB) we know

$$a_1 = \frac{1}{2}$$
$$\text{for } n > 1, \quad a_n = \frac{1}{2}\, a_{n-1}\ .$$

So

$$\begin{aligned} a_1 &= \frac{1}{2} &&= \left(\frac{1}{2}\right)^1 \\ a_2 &= \frac{1}{2}\, a_1 &&= \left(\frac{1}{2}\right)^2 \\ a_3 &= \frac{1}{2}\, a_2 &&= \left(\frac{1}{2}\right)^3 \\ a_4 &= \frac{1}{2}\, a_3 &&= \left(\frac{1}{2}\right)^4 \\ &\vdots \\ a_n &= &&\left(\frac{1}{2}\right)^n \qquad \text{for each } n \in \mathbb{N} \\ \frac{a_{n+1}}{a_n} &= \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} &&= \frac{1}{2} \qquad \text{for each } n \in \mathbb{N}. \end{aligned} \tag{2EB}$$

So we want to examine the sequence $\{s_n\}_{n=1}^{\infty}$ where

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k .$$

> When $s_n = \sum_{k=1}^{n} r^k$ for some fixed constant r,
> cancellation heaven occurs when one computes $\boxed{s_n - r s_n}$
> and we then can express s_n without using a the $\sum$-sign nor the $\ldots$-sign.

►. Let's see some cancellation heaven - in action.

$$\begin{aligned} s_n &= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^n \\ \left(\frac{1}{2}\right) s_n &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1} \end{aligned}$$

Do you see the cancellation that would occur if we take $s_n - \left(\frac{1}{2}\right) s_n$?

$$
\begin{array}{rcl}
s_n & = & \left(\frac{1}{2}\right)^1 + \cancel{\left(\frac{1}{2}\right)^2} + \ldots + \cancel{\left(\frac{1}{2}\right)^{n-1}} + \cancel{\left(\frac{1}{2}\right)^n} \\
\left(\frac{1}{2}\right) s_n & = & \cancel{\left(\frac{1}{2}\right)^2} + \cancel{\left(\frac{1}{2}\right)^3} + \ldots + \cancel{\left(\frac{1}{2}\right)^n} + \left(\frac{1}{2}\right)^{n+1}
\end{array}
$$

substract

$$
\frac{1}{2} s_n \overset{Ⓐ}{=} s_n - \left(\frac{1}{2}\right) s_n = \left(\frac{1}{2}\right)^1 - \left(\frac{1}{2}\right)^{n+1}
$$

and so we can express s_n, without using a the $\sum$-sign nor the $\ldots$-sign, as

$$
s_n = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} .
$$

Since, for each $n \in \mathbb{N}$

$$
s_n = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} \lneq \frac{\frac{1}{2} - 0}{\frac{1}{2}} = 1.
$$

Next we compute $\lim_{n\to\infty} s_n$:

$$
s_n = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} \xrightarrow[\text{since } \left|\frac{1}{2}\right| < 1]{n\to\infty} \frac{\frac{1}{2} - 0}{\frac{1}{2}} = 1 .
$$

So $\lim_{n\to\infty} s_n = 1$.
Thus Mr. Engergizer started 1 mile from 5points and

- in a finite amount of time, never quite reaches 5points (since $s_n < 1$ for each $n \in \mathbb{N}$)
- if he could go on hopping forever, he would hop 1 mile (since $\lim_{n\to\infty} s_n = 1$)
- if give enough (but finite amt. of) time, he can get arbitararily close to 5point but not quite there.

Geometric Series

A geometric series (with ratio $r \in \mathbb{R}$) is a series of the form

$$
\sum ar^n
$$

where $a \in R$ but $a \neq 0$. (Here, think of r and a as fixed numbers). A geometric series

$$
\sum r^n \text{ is } \begin{cases} \text{convergent} & \text{when } |r| < 1 \\ \text{divergent} & \text{when } |r| \geq 1 \, ; \end{cases} \tag{GeoSeries}
$$

to show this, just find the n^{th} partial sum s_n of a geometric series and use your sequence-knowledge that

$$
\lim_{n\to\infty} r^n \begin{cases} \text{diverges to } \infty & \text{if } r > 1 \\ = 1 & \text{if } r = 1 \\ = 0 & \text{if } |r| < 1 \\ \text{DNE} & \text{if } r \leq -1 \, . \end{cases} \tag{GeoSeq}
$$

How to detect a geometric series.

Let $\sum a_n$ be a series with $a_n \neq 0$ for each n. Then

$$
\left[\sum a_n \text{ is a geometric series with ratio } r \neq 0 \right] \iff \left[\frac{a_{n+1}}{a_n} = r \quad \text{for each } n \right] .
$$

Good old algebra shows this. Note if $\frac{a_{n+1}}{a_n} = r$ for $n = 0, 1, 2, \ldots$, then $a_n = r^n a_0$ for $n = 0, 1, 2, \ldots$ (see (2EB)).

Ex2. Determine if

$$\sum_{n=17}^{\infty} 5\,\frac{(-2)^n}{3^{2n+1}}$$

is a geometric series. If the series is a geometric series, then find its ratio.

Soln. The n^{th}-term a_n and the $(n+1)^{\text{st}}$-term a_{n+1} are (beware of algebra when finding a_{n+1}, common source of errors)

$$a_n = 5\,\frac{(-2)^n}{3^{2n+1}} = \frac{5\,(-2)^n}{3^{2n+1}} \qquad \text{and} \qquad a_{n+1} = \frac{5\,(-2)^{n+1}}{3^{2(n+1)+1}} = \frac{5\,(-2)^{n+1}}{3^{2n+3}}\,.$$

Let's compute.

$$\frac{a_{n+1}}{a_n} = \frac{a_{n+1}}{1}\,\frac{1}{a_n} = \frac{5\,(-2)^{n+1}}{3^{2n+3}}\,\frac{3^{2n+1}}{5\,(-2)^n} \overset{\text{Ⓐ}}{=} \underbrace{\frac{5}{5}\,\frac{(-2)^{n+1}}{(-2)^n}\,\frac{3^{2n+1}}{3^{2n+3}}}_{\text{group together like terms}} \overset{\text{Ⓐ}}{=} \frac{(-2)^1}{3^2} = \frac{-2}{9}\,.$$

Thus $\sum 5\,\frac{(-2)^n}{3^{2n+1}}$ is a geometric series with ratio $r = \frac{-2}{9}$.

Ex3. Consider the series $\sum_{n=17}^{\infty} \frac{5\,(-2)^n}{3^{2n+1}}$ from Example 2 along with its partial sums

$$s_n := \sum_{k=17}^{n} \frac{5\,(-2)^k}{3^{2k+1}} \qquad \text{for } n \geq 17\,.$$

3.1. Express s_n without using a $\ldots$-sign or $\sum$-sign.

3.2. Find $\lim_{n\to\infty} s_n$, if the limit exists.

3.3. Determine if the series converges or diverges.

3.4. If the series converges, find its sum.

Soln. From our calculations in Example 2, we already know that the series is a geometric series with ratio $r = \frac{-2}{9}$; hence, since $|r| < 1$, we know the series converges (thus we have anwered question 3.3) and the series must be of the (simplified) form $\sum a_n = \sum ar^n = \sum a\left(\frac{-2}{9}\right)^n$ for some constant a. To find this simplified form, we manipulate a_n to make it look like ar^n:

$$a_n = \frac{5\,(-2)^n}{3^{2n+1}} = \frac{5\,(-2)^n}{3\,(3^2)^n} = \frac{5}{3}\left(\frac{-2}{9}\right)^n\,;$$

thus,

$$\sum_{n=17}^{\infty} \frac{5\,(-2)^n}{3^{2n+1}} = \sum_{n=17}^{\infty} \frac{5}{3}\left(\frac{-2}{9}\right)^n \quad \text{and} \quad s_n = \sum_{k=17}^{n} \frac{5}{3}\left(\frac{-2}{9}\right)^k \overset{\text{Ⓐ}}{=} \frac{5}{3}\sum_{k=17}^{n}\left(\frac{-2}{9}\right)^k\,.$$

As in Example 1, let's find an expression for $s_n - rs_n$, which results in a *cancellation heaven*.

$$s_n = \frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} + \left(\frac{-2}{9}\right)^{18} + \ldots + \left(\frac{-2}{9}\right)^{n-1} + \left(\frac{-2}{9}\right)^{n}\right]$$

$$\left(\frac{-2}{9}\right)s_n = \frac{5}{3}\left[\left(\frac{-2}{9}\right)^{18} + \left(\frac{-2}{9}\right)^{19} + \ldots + \left(\frac{-2}{9}\right)^{n} + \left(\frac{-2}{9}\right)^{n+1}\right]$$

substract

$$\left(1 - \left(\frac{-2}{9}\right)\right)s_n \overset{\text{Ⓐ}}{=} s_n - \left(\frac{-2}{9}\right)s_n = \frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} - \left(\frac{-2}{9}\right)^{n+1}\right]$$

and so

$$s_n \overset{\text{Ⓐ}}{=} \frac{\frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} - \left(\frac{-2}{9}\right)^{n+1}\right]}{1 - \left(\frac{-2}{9}\right)} \overset{\text{Ⓐ}}{=} \frac{\frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} - \left(\frac{-2}{9}\right)^{n+1}\right]}{\frac{11}{9}}$$

$$\overset{\text{Ⓐ}}{=} \boxed{\frac{9}{11}\cdot\frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} - \left(\frac{-2}{9}\right)^{n+1}\right]} \underbrace{\xrightarrow{n\to\infty}}_{\substack{\text{for } r = \frac{-2}{9},\ \lim_{n\to\infty} r^n = 0 \text{ since } |r|<1 \\ \text{see (GeoSeq)}}} \frac{9}{11}\,\frac{5}{3}\left[\left(\frac{-2}{9}\right)^{17} - 0\right] \overset{\text{Ⓐ}}{=} \boxed{\boxed{\frac{15}{11}\left(\frac{-2}{9}\right)^{17}}}\,. \tag{3ans}$$

Note that (3ans) contains the answers since $\sum_{n=17}^{\infty} a_n = \lim_{n\to\infty}(a_{17} + a_{18} + \ldots + a_n) = \lim_{n\to\infty} s_n$.

3.1. $s_n = \frac{9}{11} \cdot \frac{5}{3} \left[\left(\frac{-2}{9}\right)^{17} - \left(\frac{-2}{9}\right)^{n+1} \right]$.
3.2. $\lim_{n\to\infty} s_n = \frac{15}{11} \left(\frac{-2}{9}\right)^{17}$.
3.3. Since the sequence of partial sums $\{s_n\}$ converges, the series $\sum a_n$ converges.
3.4. $\sum_{n=17}^{\infty} \frac{5\,(-2)^n}{3^{2n+1}} = \frac{15}{11} \left(\frac{-2}{9}\right)^{17}$

Telescoping Series

Problem. We are given a specific series $\sum_{n=1}^{\infty} a_n$ and we want to find its sum.
Goal. Find a formula (without using a ...-sign or $\sum$-sign) for s_n so we can easily compute $\lim_{n\to\infty} s_n$ where

$$s_n \;=\; a_1 + a_2 + \ldots + a_n \;=\; \sum_{k=1}^{n} a_k \qquad \text{and so} \qquad \sum_{n=1}^{\infty} a_n \;=\; \lim_{n\to\infty} s_n$$

provided the limit exists.

- For a geometric series $\sum ar^n$ we can find a formula (without a ...-sign or $\sum$-sign) for $s_n - rs_n$, which has naturally present cancellations, and then we just used simple algebra to solve for s_n.
- Now we consider a telescoping series where such a formula (without a ...-sign or $\sum$-sign) for s_n can be found directly from taking advantage of naturally present cancellations within s_n itself.

Ex4. Determine the behavior of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Check one box, and if you check the first box then fill in its blank. Justify your answer.

☐ The series converges to a finite number, in which case $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) =$ ______
☐ The series diverges to ∞, and so we write $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \infty$
☐ The series diverges to $-\infty$, and so we write $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = -\infty$
☐ The series diverges but does not diverge to $\pm\infty$, i.e. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ DNE.

Soln. Let $a_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)$ and $s_n = \sum_{k=1}^{n} a_k$. Thus we want to consider $\lim_{n\to\infty} s_n$ where

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \underbrace{\left(\frac{1}{1} - \frac{1}{2}\right)}_{k=1 \text{ term}} + \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{k=2 \text{ term}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{k=3 \text{ term}} + \ldots + \underbrace{\left(\frac{1}{n-1} - \frac{1}{n}\right)}_{k=n-1 \text{ term}} + \underbrace{\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{k=n \text{ term}} . \end{aligned}$$

Do you see some natural cancellation in the ...-formulation of s_n? If not, perhaps rewriting s_n will help.

$$s_n \;=\; \left(1 + \frac{-1}{2}\right) + \left(\frac{1}{2} + \frac{-1}{3}\right) + \left(\frac{1}{3} + \frac{-1}{4}\right) + \ldots + \left(\frac{1}{n-1} + \frac{-1}{n}\right) + \left(\frac{1}{n} + \frac{-1}{n+1}\right) .$$

cancel cancel cancel ... cancel cancel

Or perhaps you can see the natural cancellation viewing as s_n as

$$\begin{array}{llll} s_n = & 1 & + \frac{-\cancel{1}}{\cancel{2}} & \leftrightsquigarrow k = 1 \text{ term} \\ & & \swarrow & \\ + & \frac{+\cancel{1}}{\cancel{2}} & + \frac{-\cancel{1}}{\cancel{3}} & \leftrightsquigarrow k = 2 \text{ term} \\ & & \swarrow & \\ + & \frac{+\cancel{1}}{\cancel{3}} & + \frac{-\cancel{1}}{\cancel{4}} & \leftrightsquigarrow k = 3 \text{ term} \\ & & \vdots\swarrow & \\ + & \frac{+\cancel{1}}{\cancel{n-1}} & + \frac{-\cancel{1}}{\cancel{n}} & \leftrightsquigarrow k = n-1 \text{ term} \\ & & \swarrow & \\ + & \frac{+\cancel{1}}{\cancel{n}} & + \frac{-1}{n+1} & \leftrightsquigarrow k = n \text{ term} . \end{array}$$

Either viewpoint leads to

$$s_n \;=\; 1 + \frac{-1}{n+1} \quad \xrightarrow{n\to\infty} \quad 1 + 0 \;=\; 1 .$$

Thus the answer is:

[x] The series $\sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to a finite number and $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \underline{\ 1\ }$.

Ex5. Determine the behavior of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Check one box, and if you check the first box then fill in its blank. Justify your answer.

[ ] The series converges to a finite number, in which case $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underline{\qquad}$

[ ] The series diverges to ∞, and so we write $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \infty$

[ ] The series diverges to $-\infty$, and so we write $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = -\infty$

[ ] The series diverges but does not diverge to $\pm\infty$, i.e. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ DNE.

Soln. Solving the Partial Fractions Decomposition

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

leads to

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1} .$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \;=\; \sum_{n+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \;;$$

i.e., this Example 5 is just the previous Example 4 in disguise. So the answer is:

[x] The series $\sum \frac{1}{n(n+1)}$ converges to a finite number and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underline{\ 1\ }$.

Note that our Example 5 is the textbook's Example 5 on pages 587–588.

Ex6. After class, get out a pencil and try this example.

Determine the behavior of the series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$$

Check one box, and if you check the first box then fill in its blank. Justify your answer.

☐ The series converges to a finite number, in which case $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} =$ ________

☐ The series diverges to ∞, and so we write $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \infty$

☐ The series diverges to $-\infty$, and so we write $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = -\infty$

☐ The series diverges but does not diverge to $\pm\infty$, i.e. $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ DNE.

Soln. Let $a_n = \frac{2}{n(n+2)}$ and $s_n = \sum_{k=1}^{n} a_k$. Thus we want to consider $\lim_{n\to\infty} s_n$. Solving the Partial Fractions Decomposition

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

leads to

$$\frac{2}{n(n+2)} = \frac{1}{n} + \frac{-1}{n+2} \ .$$

Thus s_n takes the form

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{-1}{k+2} \right) \\ &= \underbrace{\left(\frac{1}{1} + \frac{-1}{3}\right)}_{k=1 \text{ term}} + \underbrace{\left(\frac{1}{2} + \frac{-1}{4}\right)}_{k=2 \text{ term}} + \underbrace{\left(\frac{1}{3} + \frac{-1}{5}\right)}_{k=3 \text{ term}} + \underbrace{\left(\frac{1}{4} + \frac{-1}{6}\right)}_{k=4 \text{ term}} + \ldots + \underbrace{\left(\frac{1}{n-1} + \frac{-1}{n+1}\right)}_{k=n-1 \text{ term}} + \underbrace{\left(\frac{1}{n} + \frac{-1}{n+2}\right)}_{k=n \text{ term}} . \end{aligned}$$

The natural cancellation going on is kinda hard to see so let's rewrite s_n. Since the denominators n and $n+2$ differ by 2, we will write 2 terms per line (compare with Example 4, where the denominators n and $n+1$ differed by 1 and so we wrote 1 term per line).

$$\begin{aligned} s_n = \ & \frac{+1}{1} + \frac{-1}{3} + \frac{+1}{2} + \frac{-1}{4} && \leftrightsquigarrow k = 1 \text{ and } k = 2 \text{ terms} \\ + \ & \frac{+1}{3} + \frac{-1}{5} + \frac{+1}{4} + \frac{-1}{6} && \leftrightsquigarrow k = 3 \text{ and } k = 4 \text{ terms} \\ + \ & \frac{+1}{5} + \frac{-1}{7} + \frac{+1}{6} + \frac{-1}{8} && \leftrightsquigarrow k = 5 \text{ and } k = 6 \text{ terms} \\ + \ & \frac{+1}{7} + \frac{-1}{9} + \frac{+1}{8} + \frac{-1}{10} && \leftrightsquigarrow k = 7 \text{ and } k = 8 \text{ terms} \\ \vdots \\ + \ & \frac{+1}{n-3} + \frac{-1}{n-1} + \frac{+1}{n-2} + \frac{-1}{n} && \leftrightsquigarrow k = n-3 \text{ and } k = n-2 \text{ terms} \\ + \ & \frac{+1}{n-1} + \frac{-1}{n+1} + \frac{+1}{n} + \frac{-1}{n+2} && \leftrightsquigarrow k = n-1 \text{ and } k = n \text{ terms} . \end{aligned}$$

Now the natural cancellation is clearer; indeed,

$$
\begin{array}{lcccccccl}
s_n = & \frac{+1}{1} & + & \frac{-\cancel{1}}{\cancel{3}} & + & \frac{+1}{2} & + & \frac{-\cancel{1}}{\cancel{4}} & \rightsquigarrow k=1 \text{ and } k=2 \text{ terms} \\
+ & \frac{+\cancel{1}}{\cancel{3}} & + & \frac{-\cancel{1}}{\cancel{5}} & + & \frac{+\cancel{1}}{\cancel{4}} & + & \frac{-\cancel{1}}{\cancel{6}} & \rightsquigarrow k=3 \text{ and } k=4 \text{ terms} \\
+ & \frac{+\cancel{1}}{\cancel{5}} & + & \frac{-\cancel{1}}{\cancel{7}} & + & \frac{+\cancel{1}}{\cancel{6}} & + & \frac{-\cancel{1}}{\cancel{8}} & \rightsquigarrow k=5 \text{ and } k=6 \text{ terms} \\
+ & \frac{+\cancel{1}}{\cancel{7}} & + & \frac{-\cancel{1}}{\cancel{9}} & + & \frac{+\cancel{1}}{\cancel{8}} & + & \frac{-\cancel{1}}{\cancel{10}} & \rightsquigarrow k=7 \text{ and } k=8 \text{ terms} \\
\vdots & & & & & & & & \\
+ & \frac{+\cancel{1}}{\cancel{n-3}} & + & \frac{-\cancel{1}}{\cancel{n-1}} & + & \frac{+\cancel{1}}{\cancel{n-2}} & + & \frac{-\cancel{1}}{\cancel{n}} & \rightsquigarrow k=n-3 \text{ and } k=n-2 \text{ terms} \\
+ & \frac{+\cancel{1}}{\cancel{n-1}} & + & \frac{-1}{n+1} & + & \frac{+\cancel{1}}{\cancel{n}} & + & \frac{-1}{n+2} & \rightsquigarrow k=n-1 \text{ and } k=n \text{ terms .}
\end{array}
$$

Thus we have

$$s_n \;=\; 1+\frac{1}{2}+\frac{-1}{n+1}+\frac{-1}{n+2} \quad \xrightarrow{n\to\infty} \quad 1+\frac{1}{2}+0+0 \;=\; \frac{3}{2}\ .$$

Thus the answer is:

[x] The series $\sum \frac{2}{n(n+2)}$ converges to a finite number and $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \underline{\ \frac{3}{2}\ }$.

> n^{th}-term test for divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

<u>Observation</u>: Let $\sum a_n$ converges, say $\sum_{n=1}^{\infty} a_n = 17$. What can you say about a_n?
Well $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = s_n - s_{n-1} \xrightarrow{n\to\infty} 17-17=0$.

So have: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
get the Test: If $\lim_{n\to\infty} a_n \neq 0$ (which includes the possiblity that $\lim_{n\to\infty} a_n$ DNE), then $\sum a_n$ diverges.
Warning: If $\lim_{n\to\infty} a_n = 0$, then it is possible that $\sum a_n$ converges and it is possible that $\sum a_n$ diverges.
for example: We know $\frac{1}{2^n} \xrightarrow{n\to\infty} 0$ and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ We know $\frac{1}{n} \xrightarrow{n\to\infty} 0$ and we will show $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.
Remark: The n^{th}-term test (for divergence) can show divergence but can NOT show convergence.

Ex7. Determine the behavior of the series

$$\sum_{n=1}^{\infty} \frac{3n^2}{5n^2+1}$$

Soln. Note

$$\lim_{n\to\infty} \frac{3n^2}{5n^2+1} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^2}}{\frac{5n^2+1}{n^2}} \lim_{n\to\infty} \frac{3}{5+\frac{1}{n^2}} = \frac{3}{5} \neq 0\ .$$

So by the n^{th}-term test (for divergence), the series $\sum_{n=1}^{\infty} \frac{3n^2}{5n^2+1}$ diverges.

Algebraic Properties

Since the convergence of a series is really just the convergence of its sequence of partial sums, the algebraic properties that hold for sequences also hold for series.

Algebraic Properies of Series

Let $\sum a_n$ and $\sum b_n$ both converge. Then

$$\begin{aligned}
\sum (a_n + b_n) &= \left(\sum a_n\right) + \left(\sum b_n\right) \\
\sum (a_n - b_n) &= \left(\sum a_n\right) - \left(\sum b_n\right) \\
\sum (k\, a_n) &= k \left(\sum a_n\right)
\end{aligned}$$

where k is any constant.

Note that these Algebraic Properies of Series imply the following.

Corollary to the Algebraic Properies of Series

Let $\sum c_n$ converges and $\sum d_n$ diverges.

$$\begin{aligned}
\sum (c_n + d_n) &\qquad \text{diverges} \\
\sum (c_n - d_n) &\qquad \text{diverges} \\
\sum (k\, d_n) &
\end{aligned}$$

where k is any constant.

Let's see why the first one holds. Let $\sum c_n$ converges and $\sum d_n$ diverges. We know that $\sum (c_n + d_n)$ either converges or diverges. Assume that $\sum (c_n + d_n)$ converges. Then by the Algebraic Properies of Series we would have that

$$\sum ((c_n + d_n) - c_n) \overset{\text{note}}{=} \sum d_n \qquad \text{converges.}$$

But we know that $\sum d_n$ diverges. A contradiction! So our assumption that $\sum (c_n + d_n)$ converges cannot hold. So $\sum (c_n + d_n)$ must diverge.

Positive Termed Series

Let $\sum a_n$ is a positive termed series (which just means that each term $a_n \geq 0$). Then $s_n \leq s_{n+1}$ and so the the sequence $\{s_n\}_n$ is $\nearrow$, i.e., is nondecreasing and so

- EITHER $\{s_n\}_n$ converges (to some finite number), in which case $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$
- OR $\{s_n\}_n$ diverges to ∞, in which case $\sum a_n$ diverges to ∞ and $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \infty$